

Deposit Insurance Pricing of The Mean-Reverting Model in An Uncertain Financial Market

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Abstract

Deposit insurance is a contract that allows banks to seek compensation for insurance companies when certain conditions are met (usually when assets are too low). In this paper, deposit insurance is regarded as a put option. The uncertain fractional differential equation plays an important role in describing the uncertain dynamic process. We assume that bank stock prices obey an uncertain fractional differential equation of the Caputo type. Theorems for the inverse uncertain distributions of extreme values is given based on the definition α -path. Due to the particularity of bank interest rates, it is assumed that interest rates obey an uncertain mean-reverting model. Based on the different compensation situations of insurance institutions, the price of deposit insurance against three cases has been given. Besides, numerical calculations are also illustrated to different parameters.

Keywords: Uncertainty theory, Insurance pricing, mean-reverting model, fractional differential equation

1. Introduction

The deposit insurance system is a financial security system, which refers to the establishment of an insurance institution by all types of deposit financial institutions that meet the requirements. Each depository institution, as an insured person, pays insurance premiums for them according to a certain deposit ratio and establishes a deposit insurance reserve. When a member institution encounters an operating crisis or faces bankruptcy, the deposit insurance institution provides financial assistance or directly pays part or all of the deposits to depositors, thereby protect the interests in depositors, maintaining bank credit, and stabilizing financial order. When we want to improve the

deposit insurance system, the most important thing is to determine the deposit insurance rate, that is, to set the price of deposit insurance.

The current mainstream deposit insurance pricing methods are divided into two categories: option pricing models and expected loss pricing models. The option pricing model was first proposed by Merton[1] in 1977. Merton believes that insurance deposits owned by depository institutions are regarded as put options. If the depository institution's assets on the maturity date are less than the liability, it will exercise its power. Otherwise, it will not exercise its power. Taking the similar isomorphic relationship between the deposit insurance contract and the put option as the main theoretical basis, using the option pricing model of Black-Scholes[2], Merton derived the calculation formula of the deposit insurance premium rate.

Based on Merton's ideas, many scholars have conducted in-depth research on deposit insurance rates for the perspective of option pricing. For example, Marcus and Shaked[3] improved Merton's article and proposed a new option pricing model. Ronn and Verma[4] assume that all liabilities are guaranteed by deposit insurance companies.

In the above work, the bank's assets are described by the bank's stock price, taking into account the random factors of the financial market. However, the real world is always in a state of uncertainty, especially in the financial sector. For the modelling of uncertainty factors, we can use two theories, probability theory and uncertainty theory. We can use either the stochastic differential equation in probability theory to describe stock prices or the uncertain differential equation in uncertainty theory to describe stock prices. And there is a premise when using probability theory to deal with uncertain factors: the probability distribution used in the calculation must be sufficiently close to the actual

frequency. That requires us to have sufficient sample data; Due to market or technical reason, it is sometimes difficult for us to obtain them in practice. At this time, we need to consult industry experts to estimate and give confidence.

In 2007, the uncertainty theory proposed by Liu[5] can rationally handle confidence, then it was improved in 2009[6] on normality, duality, subadditivity and product axioms. Uncertainty theory and probability theory are fundamentally different in a theoretical basis and practical fields. Uncertainty theory makes up for the inaccuracy of probability theory when dealing with the fact that the actual frequency cannot be known and the reliability of experts has to be adopted. It provides a new perspective and idea for the pricing research of financial derivatives. In 2009, Professor Liu proposed an uncertain stock model and gave a pricing formula for European options based on this model. Then Chen[9] gave the pricing formulas for American and Asian options. Since some stocks fluctuate near the average price in the long run, Peng and Yao[10] gave an uncertain mean-reversion model to describe the long-term stock price. Tian[11] obtained the pricing formula for barrier options based on the uncertainty mean-reverting stock model.

The price of assets in the future is not only related to the current price, but also to the price of a rather long period, time. The fractional derivatives provide an excellent tool for the description of memory and hereditary properties of the process, just in line with people's sensory intuition. Zhu[12] proposed the concept of an uncertain fractional differential equation and provided a new interest rate model as an application. Zhu [13] proved the existence and uniqueness of a solution to an uncertain fractional differential equation.

Regarding bank deposit insurance, we use Merton's research ideas and regard it as a put option. The assets of the bank can be measured by changes in stock prices. We assume that bank stock prices obey an uncertain fractional differential equation. The discount rate of traditional option prices generally defaults on the risk-free interest rate. However, because banks need to pay depositors' interest, and the difference in deposit methods and duration, as well as deposits and withdrawals, will affect the interest rate paid by banks. In the long run, the interest rate that the bank needs to pay is stable at a certain regression level; we assume that the interest rate that the bank needs to pay obeys an uncertain mean-reverting model.

2. Preliminaries

2.1. Uncertain variable and uncertain process

In this section, we will introduce some basic definitions and theorems in uncertainty theory. Uncertainty theory was founded by Liu in 2007 and refined by Liu in 2009. Refer to [6-8] to know more information about an uncertain variable, uncertain differential equation(UDE), uncertain process. We suppose that a real positive number p satisfies $0 \leq n - 1 < p \leq n$ and the n is a positive integer in this paper. The C_t is a Liu canonical process which satisfies that: (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2 .

Let \mathcal{L} be a σ -algebra on a nonempty set Γ . A set function $\mathcal{M} : \mathcal{L} \rightarrow [0,1]$ is called an uncertain measure if it satisfies the following axioms.

Axiom 1: $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ ;

Axiom 2: $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$;

Axiom 3: For every countable sequence of events $\Lambda_1, \Lambda_2, \dots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Then, the uncertainty distribution $\Phi(x)$ of an uncertain variable ξ is defined by Liu[5] as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. Correspondingly the expected value of uncertain variable ξ is

$$E[\xi] = \int_0^{\infty} \{\xi \geq x\} dx - \int_{-\infty}^0 \{\xi < x\} dx,$$

where at least one of the two integrals is finite.

Let C_t be a Liu process, f and g are two functions. Then Liu defines the uncertain differential equation(UDE)

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad (1)$$

and is said that Eq (1) has an α -paths X_t^α ($0 < \alpha < 1$) by solving the following differential equation

$$dX_t^\alpha = F(t, X_t^\alpha)dt + |G(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt, \quad (2)$$

$$X_0^\alpha = X_0,$$

where $\Phi^{-1}(\alpha)$ is an inverse standard normal uncertainty distribution, that is

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \quad (3)$$

Assume that X_t and X_t^α are the solutions and α -path of the Eq(1), respectively. Then

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha, \quad (4)$$

and the inverse uncertainty distribution of the solution X_t is

$$\psi_t^{-1}(\alpha) = X_t^\alpha. \quad (5)$$

For any time $s > 0$, the time integral $\int_0^s X_t dt$ has an inverse uncertainty distribution

$$\Phi_s^{-1}(\alpha) = \int_0^s X_t^\alpha dt, \quad (6)$$

the supremum $\sup_{0 \leq t \leq s} X_t$ has an inverse uncertainty distribution

$$\Phi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} X_t^\alpha, \quad (7)$$

the infimum $\inf_{0 \leq t \leq s} X_t$ has an inverse uncertainty distribution

$$\Phi_s^{-1}(\alpha) = \inf_{0 \leq t \leq s} X_t^\alpha. \quad (8)$$

Yao[14] proved that the expected value of X_t by

$$E[X_t] = \int_0^1 X_t^\alpha d\alpha. \quad (9)$$

2.2. UFDE with the Caputo type

Both Riemann-Liouville and Caputo type uncertain fractional differential equation(UFDE) was introduced by Zhu [14]. We discuss the following UFDE with the Caputo type in this paper. Suppose C_t is a Liu process, $0 < p < 1$, f and g are two functions. Then

$$\begin{cases} {}^c D^p X_t = F(t, X_t) + G(t, X_t) \frac{dC_t}{dt} \\ X_t^{(k)}|_{t=0} = x_k, \quad k = 0, 1, \dots, n-1 \end{cases} \quad (10)$$

is called an uncertain fractional differential equation(UFDE) of Caputo type. The solution to above equation satisfies the following integral equation

$$X_t = \sum_{k=0}^{n-1} \frac{x_k t^k}{\Gamma(p)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} F(s, X_s) ds + \frac{1}{\Gamma(p)} \int_0^t F(s, X_s) (t-s)^{p-1} dC_s. \quad (11)$$

where $\Gamma(p) = \int_0^\infty t^{p-1} \exp(-t) dt$ is the gamma function.

A special type of UFDE is given by Lu and Zhu[15]. Let p be a real positive number with $0 \leq n-1 < p \leq n$. Suppose that $b(t)$ and $\sigma(t) : [0, T] \rightarrow \mathbb{R}$ are two functions. Then the following UFDE of the Caputo type with initial conditions

$$\begin{cases} {}^c D^p X_t = aX_t + b(t) + \sigma(t) \frac{dC_t}{dt}, t \in [0, T] \\ X_t^{(k)}|_{t=0} = x_k, k = 0, 1, \dots, n-1 \end{cases} \quad (12)$$

has a solution

$$X_t = \sum_{k=0}^{n-1} x_k t^k E_{p, (k+1)}(at^p) + \int_0^t (t-s)^{p-1} E_{p,p}(a(t-s)^p) b(s) ds + \int_0^t (t-s)^{p-1} E_{p,p}(a(t-s)^p) \sigma(s) dC_s.$$

Considering the properties of the solution, Zhu [13] proved that if the coefficients $F(t, x)$ and $G(t, x)$ satisfy that

(i). Lipschitz condition:

$$|F(t, x) - F(t, y)| + |G(t, x) - G(t, y)| \leq L|x - y|, \forall x, y \in \mathbb{R}^n, t \in [0, +\infty),$$

(ii). linear growth condition:

$$|F(t, x) + G(t, x)| \leq L(1 + |x|), \forall x, y \in \mathbb{R}^n, t \in [0, +\infty),$$

where L is a positive constant, then the solution X_t in $t \in [0, +\infty)$ of UFDE of the Caputo type(6) exists and is unique.

To solve UFDE's solution numerically and then α -path is given. It is the solution to the following equation

$$\begin{cases} {}^c D^p X_t = F(t, X_t^\alpha) + |G(t, X_t^\alpha)|\Phi^{-1}(\alpha) \\ X_t^{(k)}|_{t=0} = x_k, k = 0, 1, \dots, n-1, \end{cases} \quad (13)$$

where $0 < \alpha < 1$ and $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

Lu and Zhu [15] also established the relation between UFDEs and FDEs. Let X_t and X_t^α be unique solution and α -path of (10), respectively. We have

$$\begin{cases} \mathcal{M}\{X_t \leq X_t^\alpha, \forall t \in [0, T]\} = \alpha \\ \mathcal{M}\{X_t > X_t^\alpha, \forall t \in [0, T]\} = 1 - \alpha. \end{cases}$$

The solution's inverse uncertain distribution is

$$\Psi_s^{-1}(\alpha) = X_t^\alpha, \quad (14)$$

which can be seen in [13].

3. Deposit insurance pricing

Regarding bank deposits, we use Merton's research ideas and regard it as a put option. A bank belongs to a financial institution whose assets can be measured by changes in stock prices. We assume that the bank's stock price obeys an uncertain fractional differential equation

$${}^c D^p S_t = \mu S_t + \sigma_2 \frac{dC_{2t}}{dt}. \quad (15)$$

In traditional option pricing, the discount rate generally defaults on the risk-free interest rate. However, the bank needs to pay depositor interest, and the difference in saving methods and duration, deposits, withdrawals and other factors will affect the interest rate paid by the bank. In the long run, the interest rate that the bank needs to pay is stable at a certain regression level. Hence, we assume that the interest rate that the bank needs to pay obeys an uncertain mean-reverting model

$$r_t = (m - ar_t)dt + \sigma_1 dC_{1t} \quad (16)$$

Therefore, the deposit insurance pricing model as follows

$$\begin{cases} dr_t = (m - ar_t)dt + \sigma_1 dC_{1t} \\ {}^c D^p S_t = \mu S_t + \sigma_2 \frac{dC_{2t}}{dt} \\ S_t^{(k)}|_{t=0} = s_k, \quad k = 0, 1, \dots, n - 1, \end{cases} \quad (17)$$

where $0 \leq n - 1 < p \leq n$, $m > 0$, $a > 0$, S_t is the stock price, $r_t > 0$ is the interest rate, μ is the stock drift, $\sigma_1, \sigma_2 > 0$ are the stock diffusions, and C_{1t}, C_{2t} are independent Liu processes. Solving the following differential equation

$$dr_t^\alpha = (m - ar_t^\alpha)dt + \sigma_1 \Phi^{-1}(\alpha)dt,$$

we get

$$r_t^\alpha = \frac{1}{a} (m + \sigma_1 \Phi^{-1}(\alpha))(1 - \exp(-at)) + \exp(-at) r_0,$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Hence uncertain differential equation

$$dr_t = (m - ar_t)dt + \sigma_1 dC_{1t}$$

has the α -path r_t^α . Then r_t has an inverse uncertainty distribution $\varphi_t^{-1} = r_t^\alpha$. So $\int_0^T r_t dt$ has an inverse uncertainty distribution

$$\Phi_T^{-1}(\alpha) = \int_0^T r_t^\alpha dt.$$

Calculating $\int_0^T r_t^\alpha dt$, we get

$$\begin{aligned} \Phi_T^{-1}(\alpha) = \frac{1}{a} & \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) (T + \right. \right. \\ & \left. \left. \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-aT)) \right]. \end{aligned} \quad (18)$$

Because $y = \exp(-x)$ is a monotone decreasing function of x , the $\exp(-\int_0^T r_t dt)$ has an inverse uncertainty distribution

$$\begin{aligned} \Phi_{1T}^{-1}(\alpha) &= \exp(\Phi_T^{-1}(1 - \alpha)) \\ &= \exp\left(-\int_0^T r_t^{1-\alpha} dt\right). \end{aligned} \quad (19)$$

Simultaneously, for the stock price system

$$\begin{cases} {}^c D^p S_t = \mu S_t + \sigma_2 \frac{dC_{2t}}{dt} \\ S_t^{(k)} = s_k, \quad k = 0, 1, \dots, n - 1. \end{cases} \quad (20)$$

Meanwhile, according to Eq.(12) and Eq.(5), the Eq.(20) has an α -path

$$S_t^\alpha = \sum_{k=0}^{n-1} s_k t^k E_{p,(k+1)} \left((\mu + \sigma_2 \Phi^{-1}(\alpha)) t^p \right).$$

Hence the S_t has an inverse uncertainty distribution

$$Y_t^{-1}(\alpha) = \sum_{k=0}^{n-1} s_k t^k E_{p,(k+1)} \left((\mu + \sigma_2 \Phi^{-1}(\alpha)) t^p \right).$$

Theorem 3.1. Let X_t and X_t^α are the unique solution and α -path, respectively, for the following UFDE

$$\begin{cases} {}^c D^p X_t = F_1(t, X_t) + G_1(t, X_t) \frac{dC_t}{dt} \\ X_t^{(k)} \Big|_{t=0} = x_k, \quad k = 0, 1, \dots, n-1. \end{cases}$$

Let Y_t and Y_t^α are the unique solution and α -path, respectively, for the following UDE

$$dY_t = F_2(t, Y_t)dt + G_2(t, Y_t)dC_t.$$

In addition, X_t and Y_t are independent uncertain processes $f(x)$ and $g(x)$ are decreasing functions of x with $f(x) \geq 0$ and $g(x) \geq 0$. Then the inverse uncertain distribution of the supremum $\sup_{0 \leq t \leq s} f(X_t)g(Y_t)$ is

$$\psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} f(X_t^{1-\alpha})g(Y_t^{1-\alpha}). \quad (21)$$

Proof. We firstly set

$$\Lambda_1^+ = \{ \gamma | f(X_t(\gamma)) \leq f(X_t^\alpha(\gamma)), g(Y_t(\gamma)) \leq g(Y_t^\alpha(\gamma)), \forall t \},$$

$$\Lambda_1^- = \{ \gamma | f(X_t(\gamma)) > f(X_t^\alpha(\gamma)), g(Y_t(\gamma)) > g(Y_t^\alpha(\gamma)), \forall t \},$$

$$\Lambda_2^+ = \{ \gamma | f(X_t(\gamma))g(Y_t(\gamma)) \leq f(X_t^\alpha(\gamma))g(Y_t^\alpha(\gamma)), \forall t \},$$

$$\Lambda_2^- = \{ \gamma | f(X_t(\gamma))g(Y_t(\gamma)) > f(X_t^\alpha(\gamma))g(Y_t^\alpha(\gamma)), \forall t \},$$

$$\Lambda_3^+ = \left\{ \gamma \mid \begin{array}{l} \sup_{0 \leq t \leq s} f(X_t(\gamma))g(Y_t(\gamma)) \leq \\ \sup_{0 \leq t \leq s} f(X_t^\alpha(\gamma))g(Y_t^\alpha(\gamma)) \end{array} \right\},$$

$$\Lambda_3^- = \left\{ \gamma \mid \begin{array}{l} \sup_{0 \leq t \leq s} f(X_t(\gamma))g(Y_t(\gamma)) \\ > \sup_{0 \leq t \leq s} f(X_t^\alpha(\gamma))g(Y_t^\alpha(\gamma)) \end{array} \right\},$$

It is that $\Lambda_1^+ \subset \Lambda_2^+ \subset \Lambda_3^+$. Besides, $\Lambda_1^- \subset \Lambda_2^- \subset \Lambda_3^-$. Then, we derive

$$\begin{aligned} \mathcal{M}\{\Lambda_3^+\} &\geq \mathcal{M}\{\Lambda_2^+\} \geq \mathcal{M}\{\Lambda_1^+\} \\ &\geq \mathcal{M}\{X_t > X_t^\alpha, Y_t > Y_t^\alpha, \forall t\} \\ &= \mathcal{M}\{X_t > X_t^\alpha, \forall t\} \wedge \mathcal{M}\{Y_t > Y_t^\alpha, \forall t\} = \\ &1 - \alpha. \end{aligned} \quad (22)$$

$$\mathcal{M}\{\Lambda_3^-\} \geq \mathcal{M}\{\Lambda_2^-\} \geq \mathcal{M}\{\Lambda_1^-\}$$

$$\begin{aligned} &\geq \mathcal{M}\{X_t \leq X_t^\alpha, Y_t \leq Y_t^\alpha, \forall t\} \\ &= \{X_t \leq X_t^\alpha, \forall t\} \wedge \mathcal{M}\{Y_t \leq Y_t^\alpha, \forall t\} \\ &= \alpha \end{aligned}$$

(23)

Since the measures of Λ_2^+ and Λ_2^- satisfy that $\mathcal{M}\{\Lambda_3^+\} + \mathcal{M}\{\Lambda_3^-\} = 1 - \alpha$.

Combining the above conclusion, we obtain that

$$\mathcal{M}\{\Lambda_3^+\} = 1 - \alpha, \mathcal{M}\{\Lambda_3^-\} = \alpha.$$

That is

$$\mathcal{M}\{\sup_{0 \leq t \leq s} f(X_t)g(Y_t) \leq \sup_{0 \leq t \leq s} f(X_t^\alpha)g(Y_t^\alpha)\} = 1 - \alpha,$$

$$\mathcal{M}\{\sup_{0 \leq t \leq s} f(X_t)g(Y_t) > \sup_{0 \leq t \leq s} f(X_t^\alpha)g(Y_t^\alpha)\} = \alpha.$$

Then let $1 - \alpha$ replace α , we get

$$\mathcal{M}\{\sup_{0 \leq t \leq s} f(X_t)g(Y_t) \leq \sup_{0 \leq t \leq s} f(X_t^{1-\alpha})g(Y_t^{1-\alpha})\} = \alpha,$$

$$\mathcal{M}\{\sup_{0 \leq t \leq s} f(X_t)g(Y_t) > \sup_{0 \leq t \leq s} f(X_t^{1-\alpha})g(Y_t^{1-\alpha})\} = 1 - \alpha.$$

That is to say, the inverse uncertain distribution of $\sup_{0 \leq t \leq s} f(X_t)g(Y_t)$ is

$$\psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} f(X_t^{1-\alpha})g(Y_t^{1-\alpha}).$$

The theorem has been proved.

Theorem 3.2. Let X_t and X_t^α are the unique solution and α -path for the following UFDE

$$\begin{cases} {}^c D^p X_t = F_1(t, X_t) + G_1(t, X_t) \frac{dC_t}{dt} \\ X_t^{(k)} \Big|_{t=0} = x_k, \quad k = 0, 1, \dots, n-1. \end{cases}$$

Let Y_t and Y_t^α are the unique solutions as well as α -path for the following UDE

$$dY_t = F_2(t, X_t)dt + G_2(t, X_t)dC_t.$$

In addition, X_t and Y_t are independent uncertain processes $f(x)$ and $g(x)$ are increasing functions of x with $f(x) \geq 0$ and $g(x) \geq 0$. Then the IUD of the supremum $\sup_{0 \leq t \leq s} f(X_t)g(Y_t)$ is

$$\psi_s^{-1}(\alpha) = \sup_{0 \leq t \leq s} f(X_t^\alpha)g(Y_t^\alpha). \quad (24)$$

Proof. The proof is similar to that of **Theorem 3.1.**

3.1. Analog European put option

When the regulatory authority reviews bank assets at regular intervals and the assets are found to be less than the liabilities, the insurance contract will be an effect. Otherwise, it will not take effect. At this point, we can regard the insurance contract as a European put option. Among them, bank assets are estimated by stock prices, and liabilities K are recorded as the strike price, which is a constant. The f_p indicates the deposit insurance contract price.

At the $t = T$, the bank's income is $(K - S_T)^+$, and at the $t = 0$, the bank's income is

$$\exp\left(-\int_0^T r_t dt\right) (K - S_T)^+.$$

Then, we get bank net income at $t = 0$

$$-f_p + \exp\left(-\int_0^T r_t dt\right) (K - S_T)^+.$$

In the same way, we can get the net income of the insurance formula at $t = 0$

$$f_p - \exp\left(-\int_0^T r_t dt\right) (K - S_T)^+.$$

From the principle of fair pricing, we get insurance pricing formula

$$f_p = E\left[\exp\left(-\int_0^T r_t dt\right) (K - S_T)^+\right] \quad (25)$$

Theorem 3.3. When treating deposit insurance as a European put option of a strike price K, we get insurance price

$$f_p = \int_0^1 \exp(-\Phi_T^{-1}(\alpha))(K - Y_T^{-1}(\alpha))^+ d\alpha, \quad (26)$$

where

$$\Phi_T^{-1}(\alpha) = \frac{1}{a} \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left(T + \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-aT)) \right],$$

$$Y_T^{-1}(\alpha) = \sum_{k=0}^{n-1} s_k T^k E_{p,(k+1)} \left(\left(\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) T^p \right).$$

Proof. Because $(K - S_T)^+$ is decreasing function of the S_T , then $(K - S_T)^+$ has an inverse uncertainty distribution

$$\Phi_{2T}^{-1}(\alpha) = (K - Y_T^{-1}(1 - \alpha))^+.$$

Because C_{1t}, C_{2t} are independent Liu processes and $z = xy$ is increasing function of x and y for $x \geq 0$ and $y \geq 0$, then $\exp(-\int_0^T r_t dt) (K - S_T)^+$ has an inverse uncertainty distribution

$$\psi_T^{-1} = \Phi_{1T}^{-1}(\alpha) \Phi_{2T}^{-1}(\alpha)$$

$$= \exp(-\Phi_T^{-1}(1 - \alpha))(K - Y_T^{-1}(1 - \alpha))^+.$$

So

$$f_p = E\left[\exp\left(-\int_0^T r_t dt\right) (K - S_T)^+\right]$$

$$= \int_0^1 \exp(-\Phi_T^{-1}(1 - \alpha))(K - Y_T^{-1}(1 - \alpha))^+ d\alpha$$

$$= \int_0^1 \exp(-\Phi_T^{-1}(\alpha))(K - Y_T^{-1}(\alpha))^+ d\alpha, \quad (27)$$

where

$$\Phi_T^{-1}(\alpha) = \frac{1}{a} \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left(T + \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-aT)) \right],$$

$$Y_T^{-1}(\alpha) = \sum_{k=0}^{n-1} s_k T^k E_{p,(k+1)} \left(\left(\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) T^p \right).$$

According to Theorems 3.3, the algorithm to compute the analogue European put option price is given as follows.

Algorithm 1. The expected value of the analogue European put option.

Step 1: Set the parameters $a, m, \sigma_1, T, r_0, \mu, \sigma_2, K, s_k (k = 0, 1, 2 \dots)$.

Step 2: Choose a large number N according to the desired precision. Set $\alpha_i = \frac{i}{N}$.

Step 3: $i = 0, Z = 0$.

Step 4: $i \leftarrow i + 1$.

Step 5:
$$x_i = \exp\left(-\frac{1}{a}\left[\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha_i}{1-\alpha_i}\right)\left(T + \frac{1}{a} \exp(-aT)\right) - \frac{1}{a}\right] + r_0(1 - \exp(-aT))\right)$$

Step 6:
$$y_i = \max\left(0, K - \sum_{k=0}^{i-1} s_k T^k E_{p,(k+1)}\left((\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_i}{1-\alpha_i})T^p\right)\right)$$

Step 7: $Z = Z + x_i y_i$.

Step 8: if $i < N - 1$, return to step 4.

Step 9: The price of Deposit insurance is $f_p = \frac{Z}{N-1}$.

Example 1. Assume deposit insurance pricing model(17) has current price $s_0 = 35, s_1 = 2$, and the drift coefficient $\mu = 0.05$, diffusion coefficient $\sigma_1 = 0.25, \sigma_2 = 0.2$, the $r_0 = 0.08$, regression coefficients $m = 1$, constant $a = 0.5, T = 1$. The strike prices $K = 32$. According to Theorem 3.3 for such model, we can calculate out the prices for different $p(0 < p \leq 2)$. Then, we get the different price of f_{pal} shown in Table 1 by Algorithm 1.

Table 1. Analog European put option price with different order p 's.

p	0.1	0.2	0.3	0.4	0.5
f_p	2.8441	3.0094	3.1476	3.2576	3.3372
p	0.6	0.7	0.8	0.9	1.0
f_p	3.3839	3.3958	3.3718	3.3113	3.2143
p	1.1	1.2	1.3	1.4	1.5
f_p	2.0867	1.9371	1.7653	1.5771	1.3786
p	1.6	1.7	1.8	1.9	2

f_p	1.1762	0.9761	0.7839	0.6049	0.4437
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Table 1 indicates that the price of f_p increases to p in $(0,0.7]$, but decreases to p in $(0.7,1]$ or $(1,2]$. The price in $(0,1]$ is greater than $(1,2]$. Since the initial $s'(0)$ works, $p = 1$ is a special point. The price jumps(decreases) a little from $p = 1$ to $p = 1.1$.

3.2 Analog American look-back put option

Supervisors regularly check bank assets to see whether there is a moment the bank assets are less than its liabilities (deposits). When it does, the insurance contract takes effect. Otherwise, it will not take effect. At this point, we can regard the insurance as an American look-back put option. At the $t = T$, the bank's income is

$$(K - \inf_{0 \leq t \leq T} S_t)^+$$

Then at the $t = 0$, the bank's income is

$$\exp\left(-\int_0^T r_t dt\right) (K - \inf_{0 \leq t \leq T} S_t)^+$$

Therefore, at $t = 0$, we got bank's net income

$$-f_{pal} + \exp\left(-\int_0^T r_t dt\right) (K - \inf_{0 \leq t \leq T} S_t)^+$$

In the same way, we can get the net income of the insurance formula at $t = 0$

$$f_{pal} - \exp\left(-\int_0^T r_t dt\right) (K - \inf_{0 \leq t \leq T} S_t)^+$$

From the principle of fair pricing, we get insurance pricing formula

$$f_{pal} = E\left[\exp\left(-\int_0^T r_t dt\right) (K - \inf_{0 \leq t \leq T} S_t)^+\right]$$

Theorem 3.4 When treating deposit insurance as an American look-back put option of a strike price K , we got insurance price

$$f_{pal} = \int_0^1 \exp(-\Phi_T^{-1}(\alpha))(K - \inf_{0 \leq t \leq T} Y_t^{-1}(\alpha))^+ d\alpha, \tag{28}$$

where

$$\Phi_T^{-1}(\alpha) = \frac{1}{a} \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left(T + \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-aT)) \right]$$

$$\Phi_T^{-1}(\alpha) = \frac{1}{a} \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left(T + \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-aT)) \right]$$

$$Y_t^{-1}(\alpha) = \sum_{k=0}^{n-1} s_k t^k E_{p,(k+1)}((\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) t^p).$$

$$Y_t^{-1}(\alpha) = \sum_{k=0}^{n-1} s_k t^k E_{p,(k+1)}((\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) t^p).$$

Proof. Because $(K - \inf_{0 \leq t \leq T} S_t)^+$ is decreasing function of S_t , then $(K - \inf_{0 \leq t \leq T} S_t)^+$ has an inverse uncertainty distribution

$$\Phi_{2T}^{-1}(\alpha) = (K - \inf_{0 \leq t \leq T} Y_t^{-1}(1 - \alpha))^+.$$

Because C_{1t}, C_{2t} are independent Liu processes and $z = xy$ is increasing function of x and y for $x \geq 0$ and $y \geq 0$, then $\exp(-\int_0^T r_t dt) (K - \inf_{0 \leq t \leq T} S_t)^+$ has an inverse uncertainty distribution

$$\psi_{2T}^{-1} = \Phi_{1T}^{-1}(\alpha) \Phi_{2T}^{-1}(\alpha) = \exp(-\Phi_T^{-1}(1 - \alpha)) (K - \inf_{0 \leq t \leq T} Y_t^{-1}(1 - \alpha))^+.$$

So

$$f_{pal} = E \left[\exp(-\int_0^T r_t dt) (K - \inf_{0 \leq t \leq T} S_t)^+ \right] = \int_0^1 \exp(-\Phi_T^{-1}(1 - \alpha)) (K - \inf_{0 \leq t \leq T} Y_t^{-1}(1 - \alpha))^+ d\alpha, \tag{29}$$

where

According to Theorems 3.4, the algorithm to compute the analogue American look-back put option price is given as follows.

Algorithm 2. The expected value of the analogue American look-back put option.

Step 1: Set the parameters $a, m, \sigma_1, T, r_0, \mu, \sigma_2, K, s_k (k = 0, 1, 2 \dots)$.

Step 2: Choose a large number N according to the desired precision. Set $\alpha_i = \frac{i}{N}$.

Step 3: Set $i = 0, Z = 0$.

Step 4: $i \leftarrow i + 1$.

Step 5: Set $x_i = \exp\left(-\frac{1}{a} \left[\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha_i}{1-\alpha_i} \right) \left(T + \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right] + r_0(1 - \exp(-aT)) \right)$.

Step 6: Choose a large number M , Set $t_j = \frac{j}{M} T$.

Step 7: $j = 0$.

Step 8: $j \leftarrow j + 1$.

Step 9: Set $A_j = \sum_{k=0}^{n-1} s_k t_j^k E_{p,(k+1)}((\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) t_j^p)$.

Step 10: If $j = 1$, then $B_j = A_j$. Else: $B_j = \min \{A_j, B_j\}$.

Step 11: If $j < M - 1$, return step 8.

Step 12: $y_i = \max(0, K - \inf_{0 \leq t_j \leq T} \sum_{k=0}^{n-1} S_k t_j^k E_{p,(k+1)}((\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_i}{1-\alpha_i}) t_j^p)$.

Step 13: $Z = Z + x_i y_i$.

Step 14: if $i < N - 1$, return to step 4.

Step 15: The price of Deposit insurance is $f_{pat} = \frac{Z}{N-1}$.

Example 2. Assume a deposit insurance pricing(17) model has current price $s_0 = 35, s_1 = 2$, and the drift coefficient $\mu = 0.05$, diffusion coefficient $\sigma_1 = 0.25, \sigma_2 = 0$. And the $r_0 = 0.08$, the regression coefficients $m = 1$, constant $a = 0.6$, $T = 1$. The strike prices $K = 32$. According to Theorem 3.4 for such model, we can calculate out the prices for different $p(0 < p \leq 2)$. Then, we get the different price of f_{pal} shown in Table 2 by Algorithm 2.

Table 2 indicates that the price of f_{pal} increases to p in $(0,0.7)$, but decreases to p in $(1,2]$ or $(0.7,1]$. And the price in $(0,1]$ is greater than $(1,2]$. Since the initial $s'(0)$ works, $p = 1$ is a special point. The price jumps(decreases) a little from $p = 1$ to $p = 1.1$.

Table 2. Analog American look-back put option price with different order p 's.

p	0.1	0.2	0.3	0.4	0.5
f_{pal}	1.6156	1.7144	1.7961	1.8604	1.9056
p	0.6	0.7	0.8	0.9	1.0
f_{pal}	1.9306	1.9343	1.9164	1.8769	1.8158
p	1.1	1.2	1.3	1.4	1.5
f_{pal}	1.1210	1.0344	0.9361	0.8297	0.7186
p	1.6	1.7	1.8	1.9	2
f_{pal}	0.6072	0.4983	0.3953	0.3009	0.2174

3.3 Analog American put option

When the bank discovers that its assets are not enough to pay its liabilities (deposits) at any time t , it proactively applies for bankruptcy liquidation. At this time, we regard insurance as an American put

option. Let f_{pa} represent the price of this insurance contract. Then at the time 0, the net return to the bank is

$$\sup_{0 \leq t \leq T} \exp(-\int_0^T r_t dt) (K - S_t)^+.$$

Then, we got Bank net income at $t = 0$

$$-f_{pa} + \sup_{0 \leq t \leq T} \exp(-\int_0^T r_t dt) (K - S_t)^+.$$

In the same way, we can get the net income of the insurance formula at $t = 0$

$$f_{pa} - \sup_{0 \leq t \leq T} \exp(-\int_0^T r_t dt) (K - S_t)^+.$$

From the principle of fair pricing, we get Insurance pricing formula

$$f_{pa} = E \left[\sup_{0 \leq t \leq T} \exp(-\int_0^T r_t dt) (K - S_t)^+ \right].$$

Theorem 3.5. When treating deposit insurance as an American put option of a strike price K , we got the insurance price is

$$f_{pa} = \int_0^1 \sup_{0 \leq t \leq T} \exp(-\Phi_t^{-1}(\alpha)) (K - Y_t^{-1}(\alpha))^+ d\alpha, (30)$$

where

$$\Phi_t^{-1}(\alpha) = \frac{1}{a} \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left(t + \frac{1}{a} \exp(-at) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-at)) \right],$$

$$Y_t^{-1}(\alpha) = \sum_{k=0}^{n-1} S_k t^k E_{p,(k+1)} \left((\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) t^p \right).$$

Proof. Because $(K - S_t)^+$ is decreasing function of the S_t , $\exp(-\int_0^T r_t dt)$ is decreasing function of the r_t , according to theorem 3.1, the $\sup_{0 \leq t \leq T} \exp(-\int_0^T r_t dt) (K - S_t)^+$ has an inverse uncertainty distribution

$$\psi_{2t}^{-1} = \sup_{0 \leq t \leq T} \exp(-\Phi_t^{-1}(1-\alpha))(K - Y_t^{-1}(1-\alpha))^+$$

So

$$\begin{aligned} f_{pa} &= E \left[\sup_{0 \leq t \leq T} \exp(-\int_0^t r_t dt) (K - S_t)^+ \right] \\ &= \int_0^1 \sup_{0 \leq t \leq T} \exp(-\Phi_t^{-1}(1-\alpha))(K - Y_t^{-1}(1-\alpha))^+ d\alpha \\ &= \int_0^1 \sup_{0 \leq t \leq T} \exp(-\Phi_t^{-1}(\alpha))(K - Y_t^{-1}(\alpha))^+ d\alpha, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \Phi_t^{-1}(\alpha) &= \frac{1}{a} \left[\left(\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha}{1-\alpha} \right) \left(t + \frac{1}{a} \exp(-at) \right) - \frac{1}{a} \right) \right. \\ &\quad \left. + r_0(1 - \exp(-at)) \right] \\ Y_t^{-1}(\alpha) &= \sum_{k=0}^{n-1} s_k t^k E_{p,(k+1)} \left(\left(\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \right) t^p \right). \end{aligned}$$

According to Theorems 3.5, the algorithm to compute the analogue American put option price is given as follows.

Algorithm 3. The expected value of the analogue American put option

Step 1: Set the parameters $a, m, \sigma_1, T, r_0, \mu, \sigma_2, K, s_k(k = 0, 1, 2, \dots)$.

Step 2: Choose a large number N according to the desired precision. Set $\alpha_i = \frac{i}{N}$.

Step 3: Set $i = 0, Z = 0$.

Step 4: $i \leftarrow i + 1$.

Step 5: Choose a large number M , Set $t_j = \frac{j}{M}T$.

Step 6: $j = 0$.

Step 7: $j \leftarrow j + 1$.

Step 8: Set $x_{ij} = \exp\left(-\frac{1}{a} \left[\left(m + \frac{\sqrt{3}\sigma_1}{\pi} \ln \frac{\alpha_i}{1-\alpha_i} \right) \left(t_j + \frac{1}{a} \exp(-aT) \right) - \frac{1}{a} \right) + r_0(1 - \exp(-at_j)) \right]$.

Step 9: Set $y_{ij} = \max(0, K - \sum_{k=0}^{n-1} s_k t_j^k E_{p,(k+1)} \left(\left(\mu + \sigma_2 \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_i}{1-\alpha_i} \right) t_j^p \right))$.

Step 10: Let $A_{ij} = x_{ij} y_{ij}$.

Step 11: If $j = 1$, then $Z_i = A_{ij}$, Else $Z_i = \max\{Z_i, A_{ij}\}$.

Step 12: If $j < M - 1$, return step 7.

Step 13: $Z = Z + x_i y_i$.

Step 14: if $i < N - 1$, return to step 4.

Step 15: The price of Deposit insurance is $f_{pa} = \frac{Z}{N-1}$.

Example 3. Assume a deposit insurance pricing model(17) has current price $s_0 = 35, s_1 = 1$, and the drift coefficient $\mu = 0.05$, diffusion coefficient $\sigma_1 = 0.25, \sigma_2 = 0.2$, the $r_0 = 0.08$, the regression coefficients $m = 1$, constant $a = 0.9$. $T = 1$. The strike prices $K = 29$. According to Theorem 3.4 for such model, we can calculate out the prices for different $p(0 < p \leq 2)$. Then, we get the different price of f_{pa} shown in Table 3 by Algorithm 3.

Table 3. Analog American put option price with different order p 's.

p	0.1	0.2	0.3	0.4	0.5
f_{pa}	9.869	10.305	10.911	11.4440	11.879
p	0.6	0.7	0.8	0.9	1.0
f_{pa}	12.215	12.431	12.511	12.439	12.205
p	1.1	1.2	1.3	1.4	1.5
f_{pa}	10.344	9.754	9.012	8.134	7.142
p	1.6	1.7	1.8	1.9	2
f_{pa}	6.063	4.923	3.745	2.554	1.370

Table 3 indicates that the price of f_{pa} increases to p in $(0,0.8]$, but decreases to p in $(0.8,1]$ or $(1,2]$. And the price in $(0,1]$ is much greater than $(1,2]$. Since the initial $s'(0)$ works, $p = 1$ is a special point. The price jumps(decreases) a little from $p = 1$ to $p = 1.1$.

In this case, the insurance company needs to take more risks, so the price is great than Table 1 with the same p .

4. Conclusion

Based on the uncertainty theory and the α – path of UFDE involving Caputo derivative, this paper assumes that the bank's stock prices to obeying a UFDE and the discount rates to obeying a UDE. Then we obtained the pricing formulas for deposit insurance against three cases and gave the numerical example of each case. There are many factors that we haven't discussed yet. For example, banks have minimum deposit requirements, and banks' interest rates have upper and lower limits. Bank assets are not only affected by stock prices, but also by deposits and withdrawals, etc. There are still a lot of works we may do.

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