

Supercritical and Subcritical Hopf Bifurcations in Non Linear Maps

Tarini Kumar Dutta, Department of Mathematics, Gauhati University

Pramila Kumari Prajapati, Department of Mathematics, Gauhati University

Sankar Haloi, Department of Mathematics, Gauhati University

Abstract

We consider a modified form of the Brusselator model as described below.

$$\dot{x} = 1 - (b+1)x + ax^2y$$

$$\dot{y} = bx - ax^2y$$

where $a, b > 0$ are parameters and $x, y \geq 0$ are dimensionless concentrations.

We develop a general theory for evaluating Supercritical and Subcritical Hopf Bifurcation for Brusselator model and obtain the Hopf bifurcation points. We also highlight some dynamical properties and time series analysis of the bifurcation points. A few open problems are posed at the end.

Key words: Limit cycles/ Stability/ Supercritical /Subcritical Hopf Bifurcation/ Time series analysis.

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Introduction

The term Hopf Bifurcation(also known as Poincare Andronov-Hopf Bifurcation),named after Henri Poincare, Eberhard Hopf and Aleksander Andronov, refers to the local birth and death of a periodic solution from an equilibrium as a parameter crosses a critical value. It is the simplest bifurcation not just involving equilibria and therefore belongs to what is sometimes termed as dynamic bifurcation theory. Hopf bifurcation occurs when a complex conjugate pair of eigenvalues linearised about the fixed point crosses the imaginary axis of the complex plane. There are two types of Hopf bifurcation , supercritical and Subcritical Hopf bifurcation. A supercritical hopf bifurcation occurs when stable limit cycles are created about an unstable critical point whereas subcritical hopf bifurcation occurs when an unstable limit cycle is created about stable critical point.

In this paper we have studied the nullclines and have shown its direction field. Next we have shown that our model satisfies Hopf bifurcation theorem and have viewed the time evaluation and phase space graph.We have also studied the behavior of the real and imaginary eigenvalues.

The Hopf Bifurcation Theorem:

We consider the planar system

$$\frac{dx}{dt} = f(x, y; \mu); \frac{dy}{dt} = g(x, y; \mu)$$

where μ is an adjustable parameter . Suppose it has a fixed point $(x, y) = (x_0, y_0)$, which may depend on μ . Let the eigenvalues of the linearised system about the fixed point be given by $\lambda(\mu), \bar{\lambda}(\mu) = \eta \pm i\delta$. Suppose further that for a certain value of μ , say $\mu = \mu_0$ the following conditions are satisfied:

- I. **Non-hyperbolicity condition** ; conjugate pair of imaginary eigenvalues.

$$\eta(\mu_0) = 0, \delta(\mu_0) = \omega \neq 0 \quad \text{where}$$

$$\text{sign}(\omega) = \text{sign}\left[\frac{\partial g}{\partial x}\bigg|_{\mu=\mu_0}(x_0, y_0)\right]$$

- II. **Transversality condition**; the eigenvalues cross the imaginary axis with non-zero speed

$$\frac{d\eta(\mu)}{d\mu}\bigg|_{\mu=\mu_0} = d \neq 0$$

- III. **Genericity Condition:**

$$A = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16w}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + g_{yy}f_{yy}]$$

with

$$f_{xy} = \left(\frac{\partial^2 f}{\partial x \partial y}\right)\bigg|_{\mu=\mu_0}(x_0, y_0)$$

Then a unique curve of periodic solutions bifurcates from the origin into the region $\mu > \mu_0$ If $A < 0$

or $\mu < \mu_0$ if $Ad > 0$. The origin is a stable fixed point for $\mu > \mu_0$ (resp. $\mu < \mu_0$) and an unstable

fixed point for $\mu < \mu_0$ (resp. $\mu > \mu_0$) if $d < 0$ (resp. $d > 0$)

whilst the periodic solutions are stable

(resp. unstable) if the origin is unstable (resp. stable) on the side of $\mu = \mu_0$ where the periodic solutions

exist. The amplitude of the periodic orbits grows like $\sqrt{|\mu - \mu_0|}$ whilst their periods tend to $\frac{2\pi}{|w|}$ as μ tends to μ_0 .

The bifurcation is called *supercritical* if the bifurcating periodic solutions are stable, and *subcritical* if they are unstable.

If $A > 0$ the periodic orbit is unstable i.e., the bifurcation is subcritical.

If $A < 0$ the periodic orbit is stable i.e., the bifurcation is supercritical.

1. The main Study of our model.

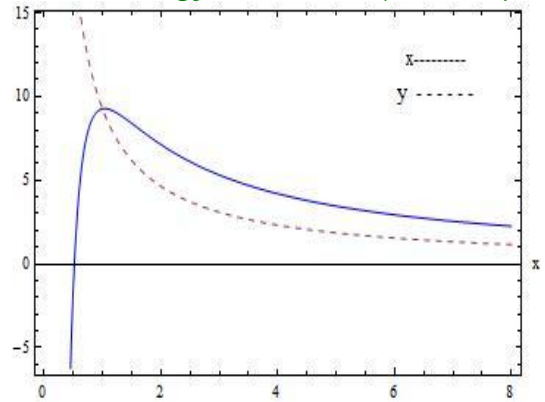
We have considered the Brusselator model

$$\begin{aligned} \dot{x} &= 1 - (b + 1)x + ax^2 y \\ \dot{y} &= bx - ax^2 y \end{aligned}$$

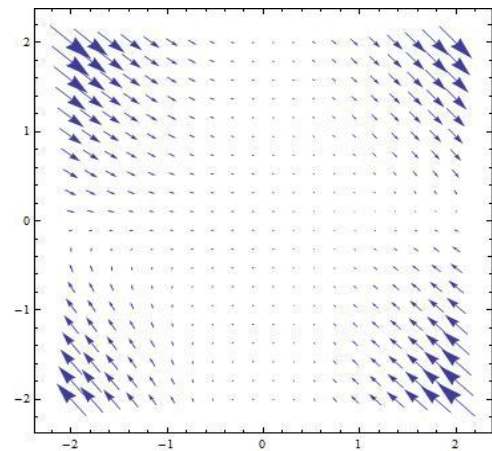
where $a, b > 0$ are controlled parameter and $x, y > 0$ are dimensionless concentration

2.1 The Nullclines and vector field :

Our model contains two variables,so it is useful to resort to the nullclines representation. These curves are defined by $\frac{dx}{dt} = 0$ on one hand and $\frac{dy}{dt} = 0$ on the other hand. Thus, the intersection of these curves corresponds to the steady state, for which we have simultaneously $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. These nullclines are sketched along with the some representative vectors. They delimit the region in the phase space where the vector field has a particular direction. In the different regions delimited by these nullclines, it is possible to determine the direction of the evolution of the system by studying the sign of $\frac{dx}{dt} = 0$ and of $\frac{dy}{dt} = 0$.



(a)



(b)

Figure.1: (a) shows nullclines and (b) direction field; where the horizontal axis shows the x-axis and the vertical axis shows the y-axis

2.2 Our Main Result :

After finding the nullclines and vector field we proceed towards linear stability analysis of our model.

The fixed point of the model is obtained as $(1, \frac{b}{a})$

The Jacobian of the system is

$$J = \begin{bmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{bmatrix}$$

$$J = \begin{bmatrix} -(b + 1) + 2axy & ax^2 \\ b - 2axy & ax^2 \end{bmatrix}$$

At the steady state $x = 1, y = \frac{b}{a}$ we have

$$J = \begin{bmatrix} -1 + b & a \\ -b & -a \end{bmatrix}$$

Eigenvalues at $(1, \frac{b}{a})$ are

$$\frac{1}{2} [(-1 - a + b) \pm \sqrt{-4a + (1 + a - b)^2}]$$

which gives complex root for some values of a and b .

Now linearising the system about the fixed point $(1, b/a)$ we get

$$\frac{dx}{dt} = 1 - (b + 1)(x + 1) + a(x + 1)^2(y + b/a)$$

$$\frac{dy}{dt} = b(x + 1) - a(x + 1)^2(y + b/a)$$

Jacobian about the origin $(0,0)$

$$J = \begin{bmatrix} -1 + b & a \\ -b & -a \end{bmatrix}$$

Hence eigenvalues at the fixed point is

$$\lambda_{1,2} = \frac{1}{2} [(-1 - a + b) \pm \sqrt{-4a + (1 + a - b)^2}]$$

For complex eigenvalues we must have

$$-4a + (1 + a - b)^2 < 0$$

This gives

$$a > 0, (1 - 2\sqrt{a}) + a < b < (1 + 2\sqrt{a}) + a$$

This implies

$$a > 0, p < b < p_1$$

where

$$p = 1 - 2\sqrt{a} + a \text{ and } p_1 = 1 + 2\sqrt{a} + a$$

For Hopf Bifurcation to occur we must have

$$-1 - a + b = 0$$

This gives

$$b = 1 + a$$

In the following table we have shown the different values of the parameters a and b where Hopf Bifurcation occurs by applying the Mathematica Software [4,5].

Table -1

A	b	p	p ₁	p < b	Eigenvalues
0.1	1.1	0.4675	1.7325	True	0.0 ± i0.316228
0.2	1.2	0.30557	2.0944	True	0.0 ± i0.447214
0.4	1.4	0.1350	2.6649	True	0.0 ± i0.632456
0.6	1.6	0.050	3.1497	True	0.0 ± i0.774597
0.8	1.8	0.01114	3.58888	True	0.0 ± i0.894427
1.0	2.0	0	4	True	0.0 ± i1
1.2	2.2	0.0091	4.3908	True	0.0 ± i1.09545
1.4	2.4	0.0335	4.7664	True	0.0 ± i1.18322
1.6	2.6	0.0701	5.1298	True	0.0 ± i1.26491
1.8	2.8	0.1167	5.4833	True	0.0 ± i1.34164
2.0	3.0	0.1716	5.8284	True	0.0 ± i1.41421
2.2	3.2	0.2335	6.1665	True	0.0 ± i1.48324
2.4	3.4	0.3016	6.4984	True	0.0 ± i1.54919

2.6	3.6	0.3750	6.8249	True	0.0 ± i1.61245
2.8	3.8	0.45335	7.14664	True	0.0 ± i1.67332
3.0	4.0	0.53589	7.46410	True	0.0 ± i1.73205
3.2	4.2	0.62229	7.7777	True	0.0 ± i1.78885
3.4	4.4	0.7122	8.0878	True	0.0 ± i1.84391
3.6	4.6	0.8053	8.3947	True	0.0 ± i1.89737
3.8	4.8	0.90128	8.69871	True	0.0 ± i1.94936
4.0	5.0	1	9	True	0.0 ± i2.0
4.2	5.2	1.1012	9.2988	True	0.0 ± i2.04939
4.4	5.4	1.2048	9.59523	True	0.0 ± i2.09762
4.6	5.6	1.3105	9.88895	True	0.0 ± i2.14476
4.8	5.8	1.4182	10.18178	True	0.0 ± i2.19089
5.0	6.0	1.52786	10.47213	True	0.0 ± i2.23607

Table showing different values of Hopf bifurcation with the increasing value of the parameter a and b with the condition $b = a + 1$

Since $a > 0$ and $b = a + 1$, let us assume the value of a as $a = 0.1 > 0$. Consequently the value of b becomes $b = 0.1 + 1 = 1.1$

Now when $a = 0.1$ and $b = 1.1$, we find

$$(1 - 2\sqrt{a} + a) < b < (1 + 2\sqrt{a} + a) .$$

this implies

$$0.46756 < b < 1.73244.$$

Clearly Hopf Bifurcation occurs at $a > 0$ and $b = 1.1$

Now we verify the different conditions of the Hopf Bifurcation theorem. [5]

i Non-hyperbolicity Condition:

$$\eta(\mu_0) = 0 \text{ and } \delta(\mu_0) = 0.31622 \neq 0 = w \text{ (say)}$$

Now we determine the sign of w .

Let

$$g_a = bx - ax^2y$$

$$\frac{\partial g}{\partial x} = b - 2axy$$

At the fixed point $x = 1, y = \frac{b}{a}$

$$\frac{\partial g}{\partial x} = b - 2a * 1 * \frac{b}{a} = b - 2b = -b$$

Hence the sign of w is negative i.e., $w = -0.31622$

ii. Transversality condition:

$$\frac{d\eta(a)}{da} = \frac{-1}{2} \neq 0$$

iii. Genericity condition :

Let

$$f(x,y) = 1 - (b+1)x + ax^2y$$

$$g(x,y) = bx - ax^2y$$

$$f_x = -(b+1) + 2axy$$

$$f_{xx} = 2ay; f_{xxy} = 2a; f_{xxx} = 0; f_{xyy} = 0;$$

$$f_{\{xy\}} = 2ax; f_{yy} = 0$$

$$g_x = b - 2axy; g_{xx} = -2ay; g_{xy} = -2ax;$$

$$g_{xxy} = 0; g_{xxy} = -2a; g_{yyy} = 0,$$

$$g_y = -ax^2, g_{yy} = 0$$

$$A = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) +$$

$$\frac{1}{16w}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + g_{yy}f_{yy}]$$

$$A = \frac{1}{16}(0 + 0 - 2a + 0) +$$

$$\frac{1}{16w}[2ax(2ay + 0) + (2ax)(-2ay + 0) + 4a^2y^2 + 0]$$

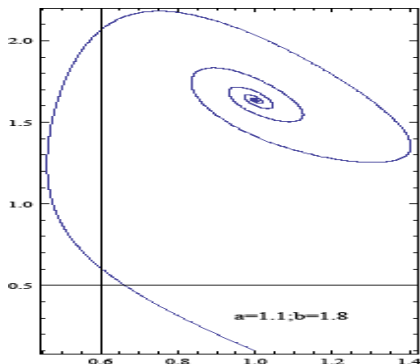
$$A = \frac{1}{16}(-0.2) + \frac{1}{16(-0.31622)}[4(1.1)^2]$$

$$A = -0.0125 - 0.9566124$$

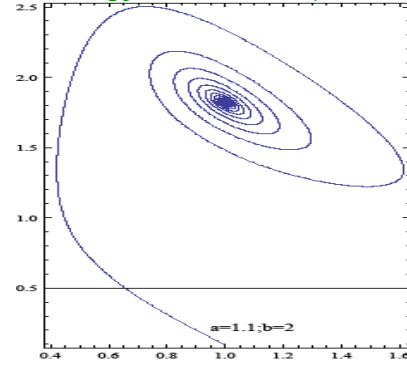
$$A = -0.969112 < 0$$

$$A = -0.969112 < 0$$

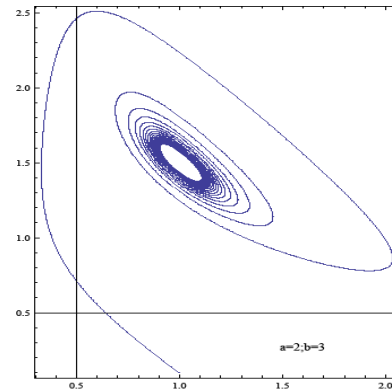
Since $A < 0$, i.e., stability coefficient is negative, the limit cycle is stable and hence our concerned model undergoes Supercritical Bifurcation.



(c)



(d)



(e)

Fig: 2. (c) and (d) shows the phase portraits when the value of a=1.1 and b=1.8, 2 whereas in (e) the value of a=2 and b=3. The horizontal line shows x-axis while the vertical line shows y-axis.

1. Linear Stability Analysis

The Brusselator model is

$$\dot{x} = 1 - (b+1)x + ax^2y$$

$$\dot{y} = bx - ax^2y$$

where $a, b > 0$ are controlled parameter and $x, y > 0$ are dimensionless concentration.

The steady state is given as

$$x = 1$$

$$y = \frac{b}{a}$$

The trace Tr and the determinant Δ of the Jacobian Matrix are:

$$Tr = b - 1 - a \text{ and } \Delta = a.$$

We study the sign of

$$\begin{aligned}
 T^2 - 4\Delta &= (-1 + b - a)^2 - 4a \\
 &= (-1 + b - a)^2 - (2\sqrt{a})^2 \\
 &= (-1 + b - a - 2\sqrt{a})(-1 + b - a + 2\sqrt{a}) \\
 &= (b - (1 + a + 2\sqrt{a}))(b - (1 + a - 2\sqrt{a})) \\
 &= (b - (\sqrt{a} + 1)^2)(b - (\sqrt{a} - 1)^2)
 \end{aligned}$$

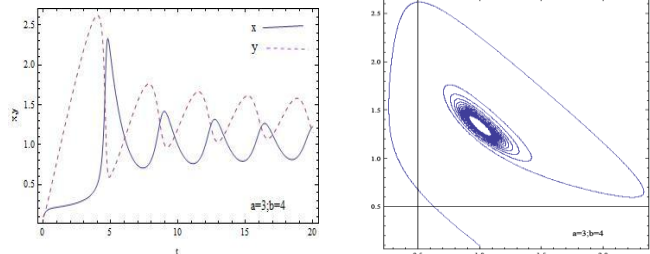
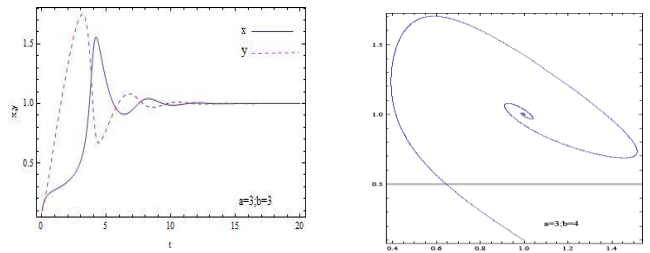
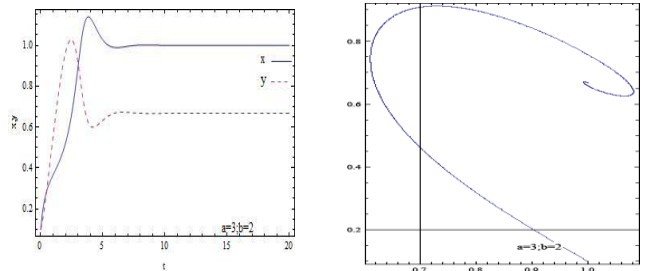
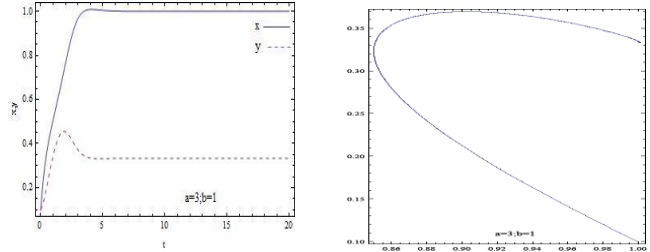
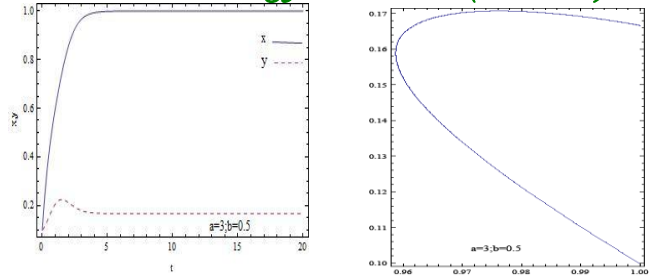
We find that the determinant Δ is always positive whereas the trace is positive if $b > (a + 1)$ otherwise negative. $Tr^2 - 4\Delta$ is positive if $(\sqrt{a} - 1)^2 < b < (\sqrt{a} + 1)^2$.

The following table shows the different behavior as a function of the parameter a and b . [3,4]

Table - 2

b		$(\sqrt{a} - 1)^2$		$(a + 1)$		$(\sqrt{a} + 1)^2$	
Tr	-	-	-	0	+	+	+
Δ	+	+	+	+	+	+	+
$Tr^2 - 4\Delta$	+	0	-	-	-	0	+
types of steady state		stable node	stable focus		unstable focus		unstable node

When parameter b increases, the steady state turns from a stable node to a stable focus, then, it loses its stability (the system then evolves towards a limit cycle) and the steady state turns from an unstable focus to an unstable node. These time evaluation and phase portrait is shown for different values of the parameter a and b where parameter b is increased keeping a constant at 3.



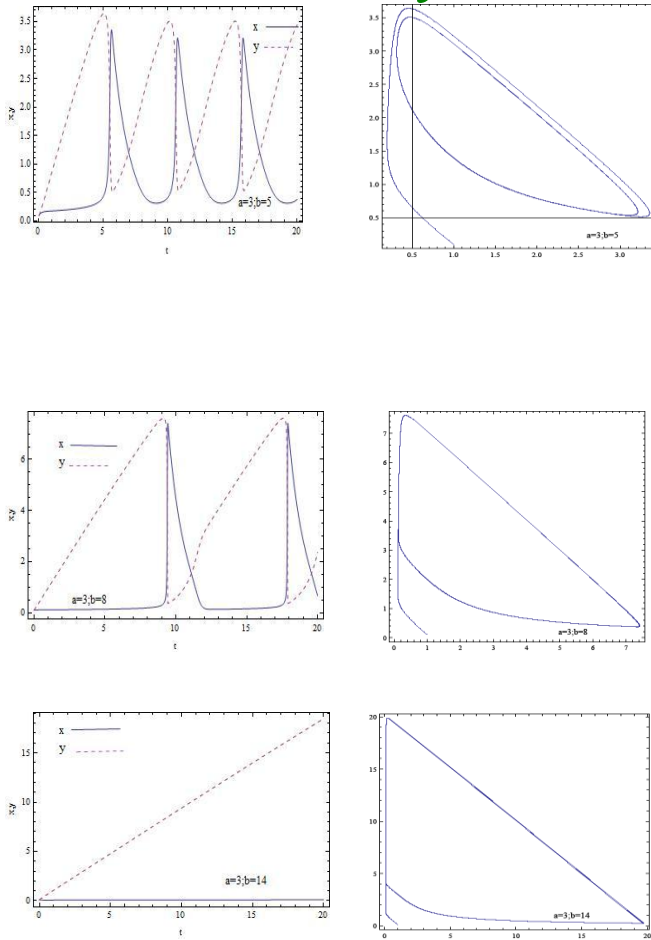


Fig: 5. Different kind of behavior obtained for the Brusselator model with parameter $a=3$ and $0.5 < b < 14$. Left panels: time evolution. Right panels: phase space. Time is plot along the abscissa and x, y is plot along the ordinate.

From the following figures we can study the behavior of our model. We have fixed a at 3 and have observed the different behavior of the time evaluation and phase portraits as the parameter b increases. As we increase b keeping a constant at 3, we find that the system evolves towards a steady state (stable node) and then evolves towards a stable focus. There is a formation of limit cycle at $b = 4$ which is quite distinct at $b = 5$. Then the system leaves its steady state (unstable focus) to reach a limit cycle at $b = 8$ and an unstable node occurs at $b = 14$. Hence this transition from an asymptotically stable equilibrium point to an unstable equilibrium point enclosed by an attracting limit cycle is supercritical Hopf Bifurcation.[2,7] Hence from the equation we have obtained the following

$$Tr = (-1 - a + b)$$

$$Determinant (\Delta) = a$$

$$Discriminant = (-1 - a + b)^2 - 4a$$

$$Real\ of\ \lambda_{(1,2)} = \frac{1}{2}(-1 - a + b)$$

$$Imaginary\ of\ \lambda_{(1,2)} = \frac{1}{2}(\pm\sqrt{-4a + (1 + a - b)^2})$$

We obtain the graph for the following

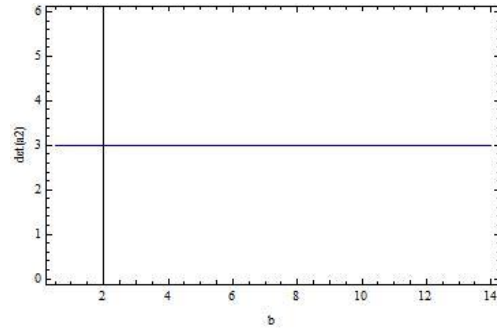


Fig: 4

In the above plot we keep a constant at 3 and vary b within the range of 0.5 to 14 and we obtain a straight line parallel to the b axis.

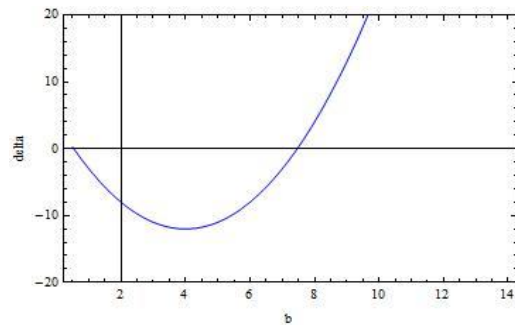


Fig: 5

In Fig: 5, we obtain a curve which shows that when a is constant at 3 and $0.53589 < b < 7.4641$ we obtain complex conjugate pairs.

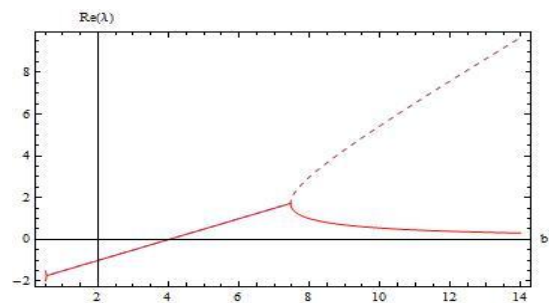


Fig: 6

The following plot is for real part of $\lambda_{\{1,2\}}$. In the Fig: 6 we find that the real of eigenvalues remains negative when b lies between 0.5 to 3.9. At $b = 4$ it the real part becomes zero and then it gives the positive value of the real part.

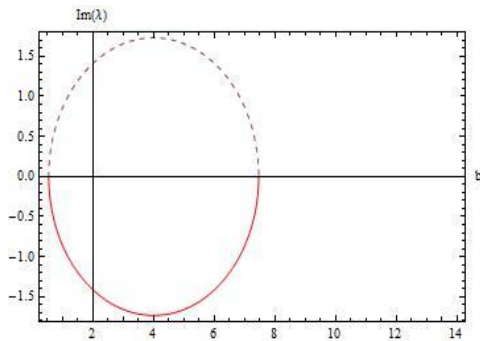


Fig:7

The above plot is for imaginary part of $\lambda_{\{1,2\}}$. In the Fig:7 we find that the imaginary part of the eigenvalues appears only when b lies between $0.535898 < b < 7.4641$ at constant a .

Open Problems:

1. Can we find the Hopf bifurcation for higher dimensional equations by applying the same technique developed in this paper.
2. Can we establish a suitable relation between Hopf Bifurcation and Period Doubling Bifurcation.
3. Can we develop a sophisticated theory in order to find Hopf Bifurcation in case of nonlinear maps.
4. Can we apply the theory of Hopf Bifurcation in order to control chaos in nonlinear system.

References

- [1] Alecea, M.R., "Introduction to Bifurcation and The Hopf Bifurcation Theorem for Planar Systems", Dynamics at the Horsetooth, M640, 2011.
- [2] Brian G. Higgins, "Nonlinear Analysis: Case Study", 2010
- [3] D. Gonze, M. Kaufman, "Theory of non-linear dynamical systems", January 6, 2014
- [4] Dutta. T.K, Das. N, "Determination of Supercritical and Subcritical Hopf Bifurcation on Two Dimensional Chaotic Model", IJASRT, ISSN2249-9954, Issue2, Vol.1 Feb 2012
- [5] Dutta. T.K., Jain. A.K., Das. N., "Analytical Process For Determination Of Hopf Bifurcations" JGRMA, ISSN 2320-5822, Vol 1, No.8, Aug 2013
- [6] Marsden, J.E. and McCracken, M., "The Hopf bifurcation and Its Application" Springer-Verlag, New York, 1976

- [7] Rousset, M.R., "Introduction To Bifurcation" Sept 16, 2005
- [8] S. Lynch, Dynamical System with Application Using Mathematica, Birkhauser, 2007
- [9] Strogatz S. H., "Nonlinear Dynamics and Chaos". Cambridge MA: Perseus, 1994
- [10] Sarma, H. K, Paul R. and Dutta, N, "Hopf Bifurcation and Normal Forms of Nonlinear Oscillators", IJRPH, ISSN0976-5840, Vol.4 No.3(2012)PP.309-321

Biographies

TARINI KUMAR DUTTA . a senior professor of the Mathematics department of Gauhati university He obtained M.Sc degree with First Class First position from Gauhati university in 1974, Ph.D. from Edinburgh University (Scotland) in 1987, Post. Doct from Edinburgh University and ETH, Switzerland in 1997-98 under Commonwealth Scholarship and Fellowship schemes respectively .After working for a few years in Arya Vidyapeeth college, Handique Girls college and Dibrugarh university in Assam state, he has been working in Gauhati university since 1978 till date . he visited UK, France, Germany, Italy, Switzerland, USA and Bangladesh for different academic programs, has published about 60 research papers, and has produced 11 Ph.D. students. His research fields are Dynamical Systems, Functional Analysis and Abstract algebras. Professor Tarini Kumar Dutta may be reached at tkdutta2001@yahoo.co.in

Pramila Prajapati is a research scholar at the department of Mathematics of Gauhati University, She obtained her M. Sc degree with First Class from Gauhati University in 2011. Her research field are Nonlinearity and Chaos. She may be reached at pramilaprajapati1987@gmail.com

Sankar Haloi is an Assistant professor in the Mathematics department of Cotton college in Assam State .He obtained his M. Sc. Degree with First Class from Gauhati university in 1995, working as an Assistant professor in Haflong Govt college during 1995-2002, Jorhat Engineering college during 2002- 2005 and is at present in Cotton college, Guwahati. His research field is Dynamical Systems. He may be reached at sankarcottonghy@rediffmail.com