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INTEGRATED METHOD OF MULTIPLE EXPERTS' DATA

Tingqing Ye[1],Yuanguo Zhu[1]

1. Department of Applied Mathematics, Nanjing University of Science and Technology, Jiangsu, China

Abstract

The uncertainty theory is a branch of mathematics for studying subjective uncertainty phenomenon. Uncertain statistics is a method for collecting and analyzing expert's experimental data by uncertainty theory. Compared to probability statistics which is based on historical data, uncertain statistics is based on experimental data. In the field of uncertain statistics, there are principle of least squares and method of moments to estimate the unknown parameters. And for multiple domain experts, Delphi method is used commonly. Delphi method requires experts' feedback on the previous round so that they can make a new judgment about altering their opinions. However, this step may not be practical. In order to make up for it, this paper puts forward two schemes which don't need experts' feedback. The feasibility and adaptability of proposed methods are described by numerical experiments.

Keywords: *uncertainty theory, uncertain statistics, Delphi method*

Introduction

Uncertainty Theory

When people describe the event of inaccurate information, they usually use “about 1.7-meter”, “approximately 15 minutes”, “high”, “short” and other such vague language. Lots of results show that these are the phenomenon of uncertain rather than random. Liu [1] declared that it is inappropriate to model belief degrees by probability theory because it may lead to counter-intuitive results. Otherwise, the fundamental condition of using probability is the requirement of observed data. Unfortunately, we are often lacking in observed data, not only for limit of some economic conditions, but also for technical reasons. In this case, we have to rely on the expert's experience and knowledge to estimate belief degrees. A belief degree represents the strength with which we believe the event will happen. From the definition, we can know that the belief degree depends heavily on the subjective judgment and personal knowledge. In order to study the phenomenon of subjective uncertainty, uncertainty theory was founded by Liu [2] in 2007. Nowadays, uncertainty theory not only develops into axiomatic mathematics branch based on normality, duality, subadditivity and product axioms, but also has achieved a series of success in the practical application.

In the aspect of basic theory, uncertain measure [2] and uncertainty space [1] were defined by Liu in 2007. And in 2010, Liu [3] proposed the concepts of uncertain set and membership function.

In the aspect of applications, in 2009, Liu [4] presented uncertain programming and successfully applied it in machine scheduling problem, vehicle routing problem and project scheduling problem. Moreover, in 2010, Liu put forward uncertain risk analysis and uncertain reliability analysis. Based on theory analysis, Liu [5] studied series system, parallel system and other different systems.

In the aspect of uncertain differential equations, in 2008, Liu [6] presented uncertain process and introduced stationary independent increment process. In 2009, Liu [7] investigated a special type of stationary independent increment process which was named as Liu process. Based on a canonical Liu process, Liu proposed Liu integral and uncertain differential equation. Uncertain differential equation was employed to model currency exchange rate by Liu, Chen and Ralescu [11].

Uncertain Statistics

Uncertain statistics is based on expert's experimental data. We could design a questionnaire survey for collecting expert's experimental data. Parametric uncertain statistics refers to the uncertainty with a known functional form but with unknown parameters. To estimate the unknown parameters, Liu [8] presented the principle of least squares and Wang and Peng [9] proposed the method of moments. When multiple experts' experimental data is available, Wang, Gao and Guo [10] recast Delphi method as a process to obtain appropriate uncertainty distributions.

Delphi method

The Delphi method originated in the early 1950s at the RAND Corporation, a California-based think-tank (Dalkey and Helmer [15], 1963). In the middle of the 20th century, when the United States government insisted on launching of the Korean War, the RAND Corporation submitted to a report to forecast that this war will be lost. The administration did not accept the results and lost the war soon after it. Named for the famed Oracle at Delphi, there have been numerous implementations and variations on the original classical Delphi method. In 1973, Harris first used it to evaluate the potential amount of mineral resources. Because some index is very difficult for experts to give a quantitative evaluation, a system analysis tool called Fuzzy Delphi Analytic Hierarchy Process (FDAHP) was proposed [16]. Because uncertain statistics is based on expert's experimental data, uncertain theory has been applied in Delphi method successfully [10].

Before using Delphi method, we should assume that group experience is more valid than individual experience. This method requires the domain experts to provide their data in two or more rounds. The steps of Delphi method are as follows: First, the participants make individual judgment independently. Then from the second round, a facilitator provides a summary of the all expert's opinions based on the previous round. And then, based on the light of the summary, the domain experts revise their earlier answers. Repeat the above process. Finally, it is believed that after these processes the experts' opinions will tend to converge to an appropriate answer.

The significance of the paper

Delphi method requires experts' feedback on the previous round so that they can make a new judgment about altering their opinions. However, this step may not be practical. Firstly, repeating the above process may be a waste of time. As long as the experts' opinions don't reach a certain standard that the facilitator sets, we have been requiring experts to change their opinions. In fact, there may be little difference between the general opinion from final round and the general opinion from the last few rounds. Secondly, sometimes, we cannot obtain all experts' opinions timely. Thirdly, the most important reason is that there is no chance for experts to provide the feedback. In other words, in some cases, it needs to generate consistent opinions after the participants make individual judgment according to individual knowledge and experience. For example, the experts estimate the valuation of a treasure. Experts put forward their own view, which means that the round of estimating the valuation of the treasure ends. In this case, there is no chance for experts to revise their first answers. Another example for judges scoring, as soon as a judger finish his scoring, we need to get the total score immediately. It is no time for judges to revise their first answers. Moreover, revising experts' own opinions will provide opportunities for fraud. In a word, in some cases, we need to generate an appropriate general opinion only based on experts' individual opinions. This paper is going to put forward two schemes to make up for the above shortcomings.

Preliminary

In convenience, we give some useful concepts at first.

Let Γ be a nonempty set and L a σ-algebra over Γ. Each element Λ in L is called an event. A number $M\{\Lambda\}$ will be assigned to each event to indicate the belief degree with which we believe Λ will happen. In order to rationally deal with belief degrees, Liu suggested the following three axioms:

Axiom 1. (Normality Axiom) $M\{\Gamma\}=1$;

Axiom 2. (Duality Axiom) $M\{\Lambda\}+M\{\Lambda^c\}=1$ for any event Λ;

Axiom 3. (Subadditivity Axiom)

$M\{\bigcup_{i=1}^{\infty}\Lambda_i\}\le\sum_{i=1}^{\infty}M\{\Lambda_i\}$ for $\Lambda_i\in\Gamma$, $i=1,2,\cdots$.

The triplet (Γ,L,M) is said to be an uncertainty space. Based on the uncertainty space, the product uncertain measure was defined by Liu [7].

Axiom 4. (Product Axiom) Let (Γ_k,L_k,M_k) be uncertainty spaces and Γ_k are nonempty sets on which M_k are uncertain measures for $k=1,2,\cdots$. The product uncertain measure M is an uncertain measure on the product σ-algebra $L_1\times L_2\times\cdots$ satisfying

$$M\{\prod_{k=1}^{\infty}\Lambda_k\}=\bigwedge_{k=1}^{\infty}M\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from L_k for $k=1,2,\cdots$, respectively.

An uncertain variable is a measurable function ξ from an uncertainty space (Γ,L,M) to the set of real numbers, i.e., for any Borel set of real numbers, the set

$$\{\xi\in B\}=\{\gamma\in\Gamma\mid\xi(\gamma)\in B\}$$

is an event.

Definition 1 (*Liu [2]*) The uncertainty distribution $\Phi(x):R\to[0,1]$ of an uncertain variable ξ is defined by

$$\Phi(x)=M\{\xi\le x\}.$$

Definition 2 (*Liu [2]*) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi]=\int_0^{+\infty} M\{\xi \ge r\}dr-\int_{-\infty}^{0} M\{\xi \ge r\}dr \quad (1)$$

provided that at least one of the two integrals is finite.

To collect expert's experiment data, we can invite one or more domain experts to complete a questionnaire about the meaning of an uncertain variable ξ like "How far from Beijing to Nanjing". Then, ask the expert to choose a possible value x (say 900km) that ξ may take and answer "How likely is ξ not larger than x ?" Denote his answer by α (say 0.7). We call (x,α) the expert's experiment data.

Repeating the above process, the following expert's experimental data are acquired from a domain expert:

$$(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_n,\alpha_n).$$

Obviously, the set of those data should meet two conditions:

$$x_1<x_2<\cdots<x_n,\ 0\le\alpha_1\le\alpha_2\le\cdots\le\alpha_n\le 1.$$

Based on the expert's experimental data, Liu suggested an empirical uncertainty distribution:

$$\Phi(x)=\begin{cases} 0, & \text{if } x<x_1 \\ \alpha_i+\dfrac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i\le x\le x_{i+1}, 1\le i<n \\ 1, & \text{if } x>x_n. \end{cases} \quad (2)$$

The empirical uncertainty distribution has an expected value

$$E[\xi]=\frac{\alpha_1+\alpha_2}{2}x_1+\sum_{i=2}^{n-1}\frac{\alpha_{i+1}-\alpha_{i-1}}{2}x_i+(1-\frac{\alpha_{n-1}+\alpha_n}{2})x_n. \quad (3)$$

Integrated Method of Multiple Experts' Data

The method is aimed at obtaining the appropriate integrated data. The main ideal is abandoning abnormal data which are greatly different from others' data. Let ξ be an uncertain variable. We first invite m experts to choose some possible values that the uncertain variable ξ may take. Each expert's possible values and the number of values can be different. The integrated method of multiple experts' data with two schemes is as follows.

Step 1. Invite m experts to provide their data (x_{ij},α_{ij}), where x_{ij} represents the j th possible value provided by the i th expert and α_{ij} represents the belief degree that $x_{ij}\le\xi$ provided by the i th expert, $i=1,2,\cdots,m$, $j=1,2,\cdots,n_i$.

Step 2. Use the i th expert's experimental data $(x_{i1},\alpha_{i1}),(x_{i2},\alpha_{i2}),\cdots,(x_{in},\alpha_{in})$ to generate the empirical uncertainty distribution $\Phi_i(x)$ of the i th domain expert, $i=1,2,\cdots,m$. Calculate the number of the possible values of ξ provided by all experts and donate the number by n, where the same values from experts are treated as one. Then the possible values of ξ from all experts are $x_1<x_2<\cdots<x_n$.

Step 3. Replenish experts' data completely by the empirical uncertainty distribution. Then, we obtain all experts' data about possible values $x_1,x_2,\cdots,x_n$ that the m experts choose.

Step 4. To generate the appropriate integrated data $(x_1,\overline{\alpha}_1),(x_2,\overline{\alpha}_2),\cdots,(x_n,\overline{\alpha}_n)$, this paper puts forward two schemes which will be introduced in the following section.

Step 5: Use the integrated data $(x_1,\overline{\alpha}_1),(x_2,\overline{\alpha}_2),\cdots,(x_n,\overline{\alpha}_n)$ to generate the empirical uncertainty distribution $\Phi(x)$ of ξ and then obtain the expected value of ξ.

Remark 1 Before asking a domain expert, none of x, α and n could be assigned a value.

Remark 2 None of us exactly know the real answer to the target problem, which ensure the significance of investigation.

Remark 3 We need to assume that experts' understanding of the target problem should not be different. When we choose the m domain experts, we should avoid the following case. If an expert really knows the target problem, obviously, his data are very close to the true value. However, his data greatly differ from others' data. If we view his data as abnormal data and delete them, the final answer by the integrated method of multiple experts' data will be unsatisfactory even wrong. To avoid above case, we need to ensure that the general opinion is closer to the true value than the extreme individual opinion when choosing experts.

Remark 4 The method is aimed at obtaining the appropriate integrated data, but we still compute the empirical uncertainty distribution and the expected value in step 5. Compute the empirical uncertainty distribution because in many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself. In addition, use the expected value because expected value is the average value of uncertain variable in the sense of uncertain measure and the expected value is a value which is convenient for us to compare, discuss and evaluate.

Scheme 1

Calculate the expected value of each empirical uncertainty distribution.

$$E_i = \frac{\alpha_{i1}+\alpha_{i2}}{2}x_1 + \sum_{j=2}^{n-1}\frac{\alpha_{ij+1}-\alpha_{ij-1}}{2}x_j + (1-\frac{\alpha_{in-1}+\alpha_{in}}{2})x_n, i=1,2,\cdots,m \quad .(4)$$

Denote

$$\overline{E} = \frac{1}{m}\sum_{i=1}^{m}E_i \, , i=1,2,\cdots,m . \qquad (5)$$

If we find the k th expert's data satisfying $\left|E_k - \overline{E}\right| > \varepsilon_1$ (ε_1 is a given value), we should delete the k th expert's data and put the number k into a collection M . In other word, we do not use the k th expert's data. We need inspect all data he provides.

Then, we take the average values of the remaining experts' opinions as the appropriate integrated data.

$$\overline{\alpha}_j = \frac{1}{m}\sum_{i\in T-M}\Phi_i(x_j), \; j=1,2,\cdots,n . \qquad (6)$$

We view all data provided by one expert as a group and compare the expected value of the group with the average expected value of all groups in the scheme 1. In other word, we obtain the appropriate integrated data by experts.

Example 1

Consider a numerical experiment. Let $m=8$, $n=10$, $\varepsilon_1=20$, $(x_1,x_2,\cdots,x_{10})=(10,20,\cdots.100)$. The experimental data provided by 8 domain experts are as follows:

A_1 :(10,0),(30,0),(40,0.1),(50,0.25),(70,0.4),(80,0.55),
(90,0.7),(100,0.85)

A_2 :(10,0),(20,0.05),(30,0.1),(50,0.35),(60,0.4),(70,0.5),
(80,0.9),(100,1)

A_3 :(10,0),(20,0.1),(30,0.15),(40,0.2),(50,0.25),(60,0.4),
(70,0.45),(80,0.5),(90,0.9),(100,1)

A_4 :(10,0.1),(20,0.15),(40,0.4),(50,0.45),(60,0.55),(70,0.6),
(80,0.65),(90,0.8),(100,0.9)

A_5 :(10,0.15),(20,0.2),(30,0.4),(40,0.45),(50,0.6),(70,0.8),
(80,0.85),(90,0.9),(100,1)

A_6 :(10,0.2),(20,0.4),(30,0.55),(40,0.75),(60,0.8),(70,0.9),
(90,1),(100,1)

A_7 :(10,0),(20,0.1),(30,0.1),(40,0.2),(50,0.25),(60,0.3),
(70,0.4),(80,0.55),(90,0.55),(100,1)

A_8 :(10,0),(20,0),(30,0),(40,0),(50,0.1),(60,0.1),(80,0.2),
(90,0.5),(100,1)

where A_i represent i th domain expert, $i=1,2,\cdots,8$.

To observe intuitively, draw the above data into the Table 1. In addition, because each expert's possible values and the number of values can be different, some data can be lacking and "-" represents the lacking data.

And then, replenish experts' data completely by the empirical uncertainty distribution. See Table 2.

Table 1. Individual Experts' Data

Experts	10	20	30	40	50	60	70	80	90	100
A_1	0	-	0	0.1	0.25	-	0.4	0.55	0.7	0.85
A_2	0	0.05	0.1	-	0.35	0.4	0.5	0.9	-	1
A_3	0	0.1	0.15	0.2	0.25	0.4	0.45	0.5	0.9	1
A_4	0.1	0.15	-	0.4	0.45	0.55	0.6	0.65	0.8	0.9

A_5	0.15	0.2	0.4	0.45	0.6	-	0.8	0.85	0.9	1
A_6	0.2	0.4	0.55	0.75	-	0.8	0.9	-	1	1
A_7	0	0.1	0.1	0.2	0.25	0.3	04.	0.55	0.9	1
A_8	0	0	0	0	0.1	0.1	-	0.2	0.5	1

Table 2. Experts' Complete Individual Data

Experts	10	20	30	40	50	60	70	80	90	100
A_1	0	0	0	0.1	0.25	0.325	0.4	0.55	0.7	0.85
A_2	0	0.05	0.1	0.225	0.35	0.4	0.5	0.9	0.95	1
A_3	0	0.1	0.15	0.2	0.25	0.4	0.45	0.5	0.9	1
A_4	0.1	0.15	0.273	0.4	0.45	0.55	0.6	0.65	0.8	0.9
A_5	0.15	0.2	0.4	0.45	0.6	0.7	0.8	0.85	0.9	1
A_6	0.2	0.4	0.55	0.75	0.775	0.8	0.9	0.95	1	1
A_7	0	0.1	0.1	0.2	0.25	0.3	04.	0.55	0.9	1
A_8	0	0	0	0	0.1	0.1	0.15	0.2	0.5	1

Based on formula (4) and (5), compute the expected value "E_j" of each empirical uncertainty distribution. See Table 3. And in the Table 3, "ΔE" represents $\left|E_k - \overline{E}\right|$ which is convenient to compare.

In Table 3, because $\overline{E}_6, \overline{E}_8 > \varepsilon_1$, we should delete the 6th and 8th experts' data. Then, use formula (6) to calculate average values of the remaining experts' opinions which is considered as the appropriate integrated data. See Table 4.

Table 3. Expected Values

Experts	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
E_j	72.5	60.25	65.5	56.25	45.25	32.75	67	84.5
$\overline{E}$	60.5							
ΔE	12	0.25	5	4.25	15.25	27.75	6.5	24

Table 4. Appropriate Integrated Data

x	10	20	30	40	50	60	70	80	90	100
$\overline{\alpha}_j$	0.042	0.1	0.171	0.263	0.358	0.446	0.525	0.667	0.858	0.958

Based on the appropriate integrated data in Table 4, we can get the empirical uncertainty distribution

$$\Phi(x)=\begin{cases} 0, & if\ \ x<10 \\ 0.042+0.0058(x-10), & if\ \ 10\le x\le 20 \\ 0.100+0.0071(x-20), & if\ \ 20\le x\le 30 \\ 0.171+0.0092(x-30), & if\ \ 30\le x\le 40 \\ 0.263+0.0095(x-40), & if\ \ 40\le x\le 50 \\ 0.358+0.0088(x-50), & if\ \ 50\le x\le 60 \\ 0.446+0.0079(x-60), & if\ \ 60\le x\le 70 \\ 0.525+0.0142(x-70), & if\ \ 70\le x\le 80 \\ 0.667+0.0191(x-80), & if\ \ 80\le x\le 90 \\ 0.858+0.0100(x-90), & if\ \ 90\le x\le 100 \\ 1, & if\ \ x>100. \end{cases}$$

And the expected value of the empirical uncertainty distribution Φ is 61.12. In conclusion, by the scheme 1, we get $E[x]=61.12$.

Scheme 2

View the possible values on the same x_j provided by all experts as a group. Choose the group whose variance is more than a given level ε_2 and use the deviation reduction method on each group. The deviation reduction method is introduced at the following part.

Deviation Reduction Method

Definition 3 The deviation d of a group of values $\alpha_1,\alpha_2,\cdots,\alpha_m$ is defined by

$$d=\frac{1}{m}\sum_{i=1}^{m}(\alpha_i-\overline{\alpha})^2 \tag{7}$$

where $\overline{\alpha}=\frac{1}{m}\sum_{i=1}^{m}\alpha_i$.

We should assume that the deviation of the group of values $\alpha_1,\alpha_2,\cdots,\alpha_m$ is larger than a given value $\varepsilon_2>0$. Then, we apply the deviation reduction method to the group of values, which means that get the average value after the deviation is reduced to a certain level ε_2 by abandoning the abnormal value. The specific steps are as follows.

Step 1. Let $T^{(1)}=\{1,2,\cdots,m\}$. Then, we compute

$$\overline{\alpha}^{(1)}=\frac{1}{m}\sum_{i=1}^{m}\alpha_i, \tag{8}$$

$$d^{(1)}=\frac{1}{m}\sum_{i=1}^{m}(\alpha_i-\overline{\alpha}^{(1)})^2. \tag{9}$$

Let

$$A^{(1)}=\sum_{i=1}^{m}\alpha_i^2, \tag{10}$$

so

$$d^{(1)}=\frac{1}{m}A^{(1)}-(\overline{\alpha}^{(1)})^2. \tag{11}$$

Step 2. $k\leftarrow 1$.

Step 3. Set

$$t^{(k)}=\arg\max_{i\in T^{(k)}}\left\{\left|\alpha_i-\overline{\alpha}^{(k)}\right|\right\}, \tag{12}$$

$$T^{(k+1)}=T^{(k)}-\left\{t^{(k)}\right\}, \tag{13}$$

$$A^{(k+1)}=A^{(k)}-\alpha_{t^{(k)}}^2, \tag{14}$$

$$\overline{\alpha}^{(k+1)}=\frac{1}{m-k}\sum_{i\in T^{(k+1)}}\alpha_i=\frac{(m-k+1)\overline{\alpha}^{(k)}-\alpha_{t^{(k)}}}{m-k}, \tag{15}$$

$$d^{(k+1)}=\frac{A^{(k+1)}}{m-k}-(\overline{\alpha}^{(k+1)})^2. \tag{16}$$

Step 4. $k\leftarrow k+1$. If $d^{(k)}>\varepsilon_2$, go to step 3.

Step 5. The appropriate average value is $\overline{\alpha}^{(k)}$.

In the above method, the introduction of $A^{(k)}$ is aimed at reducing the amount of computation, because the data in k th iterations can be used to calculate the deviation in $(k+1)$ th iteration.

Theorem By the deviation reduction method, the deviation can reach the given level $\varepsilon_2>0$ in the most m iterations.

Proof: We first prove that after every step, the deviation is descent.

From the equation (15), there exists a number $a\neq 0$ such that

$$\overline{\alpha}^{(k+1)} = \overline{\alpha}^{(k)} + a. \quad (17)$$

From the equation (12), we can draw the conclusion that

$$\left|\alpha_{t^{(k)}} - \overline{\alpha}^{(k)}\right| > \left|\alpha_i - \overline{\alpha}^{(k)}\right| \quad (18)$$

for arbitrary $i \in T^{(k+1)}$. So,

$$(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})^2 > \frac{1}{m-k}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2, \quad (19)$$

and

$$d^{(k+1)} - d^{(k)}$$
$$= \frac{1}{m-k}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k+1)})^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k)}}(\alpha_i - \overline{\alpha}^{(k)})^2$$
$$= \frac{1}{m-k}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)} - a)^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k)}}(\alpha_i - \overline{\alpha}^{(k)})^2$$
$$-\frac{1}{m-k+1}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})^2$$
$$= \frac{1}{(m-k)(m-k+1)}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k+1)}}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})$$
$$-\frac{2a}{m-k}\left(\sum_{i\in T^{(k+1)}}\alpha_i - \sum_{i\in T^{(k+1)}}\overline{\alpha}^{(k)}\right) + a^2$$
$$= \frac{1}{(m-k)(m-k+1)}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k+1)}}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})$$
$$-\frac{2a}{m-k}[(m-k)\alpha^{(k+1)} - (m-k)\overline{\alpha}^{(k)}] + a^2$$
$$= \frac{1}{(m-k)(m-k+1)}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k+1)}}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})$$
$$-\frac{2a}{m-k}(\alpha^{(k+1)} - \overline{\alpha}^{(k)}) + a^2$$
$$= \frac{1}{(m-k)(m-k+1)}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k+1)}}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})$$
$$-2a^2 + a^2$$
$$= \frac{1}{(m-k)(m-k+1)}\sum_{i\in T^{(k+1)}}(\alpha_i - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}\sum_{i\in T^{(k+1)}}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)}) - a^2$$
$$< \frac{1}{m-k+1}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})^2 - \frac{1}{m-k+1}(\alpha_{t^{(k)}} - \overline{\alpha}^{(k)})^2 - a^2$$
$$= -a^2 < 0.$$

Thus, $d^{(k+1)} < d^{(k)}$. In other word, when abandoning a value, the deviation is decent. And in the worst case, after abandoning $m-1$ values in m iterations, the deviation $d^{(m)} = 0 < \varepsilon_2$. So, by the deviation reduction method, the deviation can reach the given level $\varepsilon_2(\varepsilon_2 > 0)$ in the most m iterations.

Example 2

Still use the experts' data in example 1. And let $\varepsilon_2 = 0.02$.

Based on the formulas (8), (9) and (11), we compute $\overline{\alpha}_j^{(1)}$ and $d_j^{(1)}$, $j = 1,2,\cdots,10$. The results can be seen in Table 5.

Table 5. Individual Experts' Data

x	10	20	30	40	50	60	70	80	90	100
$\overline{\alpha}_j^{(1)}$	0.056	0.125	0.197	0.291	0.378	0.447	0.525	0.644	0.831	0.969
$d_j^{(1)}$	0.0059	0.015	0.03382	0.04874	0.0424	0.04507	0.05	0.05465	0.02309	0.00309

Because $d_3^{(1)}, d_4^{(1)}, d_5^{(1)}, d_6^{(1)}, d_7^{(1)}, d_8^{(1)}, d_9^{(1)}$ are all larger than $\varepsilon_2(\varepsilon_2 = 0.02)$, we need apply the deviation reduction method on above groups. The result is showed in Table 6. The "$\overline{\alpha}_j^{(4)}$" and "$d_j^{(4)}$" represent the average values and deviation respectively after 4 iterations with the deviation reduction method. The sign "-" is on behalf of the abnormal value which has been abandoned.

Because $\max_{1\le j\le 10} d_j^{(4)} = d_7^{(4)} = 0.0198 < 0.02 = \varepsilon_2$, the $\overline{\alpha}_j^{(4)}$ are the integrated data, $j = 1,2,\cdots,10$. And, draw the appropriate integrated data into Table 7.

Table 6. Experts' Data after the Deviation Reduction Method

Experts	10	20	30	40	50	60	70	80	90	100
A_1	0	0	0	0.1	0.25	0.325	0.4	0.55	0.7	0.85
A_2	0	0.05	0.1	0.225	0.35	0.4	0.5	-	0.95	1
A_3	0	0.1	0.15	0.2	0.25	0.4	0.45	0.5	0.9	1
A_4	0.1	0.15	0.273	0.4	0.45	0.55	0.6	0.65	0.8	0.9
A_5	0.15	0.2	0.4	-	-	-	0.8	0.85	0.9	1
A_6	0.2	0.4	-	-	-	-	-	-	1	1
A_7	0	0.1	0.1	0.2	0.25	0.3	04.	0.55	0.9	1
A_8	0	0	0	0	0.1	0.1	-	-	-	1
$\overline{\alpha}_j^{(4)}$	0.0563	0.125	0.1464	0.1875	0.275	0.3458	0.525	0.62	0.8786	0.9688
$d_j^{(4)}$	0.0059	0.015	0.0183	0.0150	0.0115	0.0184	0.0198	0.0156	0.0085	0.0031

Table 7. Appropriate Integrated Data

x	10	20	30	40	50	60	70	80	90	100
$\overline{\alpha}_j$	0.0563	0.125	0.1464	0.1875	0.275	0.3458	0.525	0.62	0.8786	0.9688

From the appropriate integrated data, we get the following empirical uncertainty distribution

$$\Phi(x)=\begin{cases} 0, & \text{if } x<10 \\ 0.0563+0.00687(x-10), & \text{if } 10\le x\le 20 \\ 0.1250+0.00214(x-20), & \text{if } 20\le x\le 30 \\ 0.1464+0.00411(x-30), & \text{if } 30\le x\le 40 \\ 0.1857+0.00875(x-40), & \text{if } 40\le x\le 50 \\ 0.2750+0.00708(x-50), & \text{if } 50\le x\le 60 \\ 0.3458+0.01792(x-60), & \text{if } 60\le x\le 70 \\ 0.5250+0.00950(x-70), & \text{if } 70\le x\le 80 \\ 0.8300+0.02586(x-80), & \text{if } 80\le x\le 90 \\ 0.8786+0.00902(x-90), & \text{if } 90\le x\le 100 \\ 1, & \text{if } x>100. \end{cases}$$

And the expected value of the empirical uncertainty distribution Φ is 63.8415. In conclusion, by the scheme 2, we get $E[x]=63.8415$.

Conclusion

From above two examples, we can find some useful information. In the aspect of calculation, scheme 1 is more convenient than scheme 2. However, in the aspect of result, scheme 1 may be not more accurate than scheme 2. Firstly, in example 1, we delete the 6th and 8th experts' data. Also, in example 2, we delete 5 data of 6th expert and 3 data of 8th expert in Table 6. It shows that the abnormal values abandoned by scheme 2 mainly are provided by experts whose data are abandoned by scheme 1 from above two examples. So, there is little difference between the expected values of the two schemes. Secondly, in Table 6, we can find that we don't get rid of all of the 6th and 8th experts' data and some are useful, which shows the roughness of scheme 1. Compared with scheme 1, we delete single data rather than all data of one expert, which shows that scheme 1 disposes data more roughly than scheme 2. Thirdly, in Table 6, some other experts' data are deleted and it shows that scheme 2 deals with data in the round.

Based on the Delphi method, the integrated method in this paper is a new way which is aimed at making up for some shortcomings of Delphi method in some cases to obtain the appropriate integrated data when dealing with multiple experts' individual experimental data that the domain experts only can provide, and the two examples show the method is effective.

Obviously, if we apply the Delphi method and the integrated method to the same meaning of an uncertain variable ξ, we can get two different empirical uncertainty distributions and their expected values as their predicted values. And if we compare their predicted values with true value, it is very likely that the predicted value of the Delphi method is closer to the true value than the integrated method. The case is acceptable because the integrated method is only based on experts' individual experimental data and the Delphi method is not only based on individual data but also experts' feedback. Moreover, in most cases, there is small difference between two predicted values and the error from the integrated method usually in the acceptable range, which shows the practicability of the integrated method. Especially when dealing with the case that there is no chance for experts to revise their first answers, this method is particularly useful.

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