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ADVANCED ORTHOGONAL TRANSFORM ALGORITHM FOR OPTIMAL ANALYSIS AND DESIGN OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

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Abstract

A method for analysis and design of distributed parameter systems using the orthogonal functions is presented in this paper. In classical control theory, control system may be modeled as systems of lumped parameters or as distributed parameters systems. The distributed parameters systems are modeled by partial differential equations. Since 1970s, orthogonal functions and its transforms such as Walsh, block pulse and Haar have been developed and used for solving analysis and design of control system including nonlinear distributed parameter systems. The algorithm adopted in this paper is that of approximating parameters of nonlinear distributed parameter systems by converting and transforming a partial differential equation into a simple algebraic equation. The applied method is very useful to analysis of nonlinear distributed parameter systems and it is superior to conventional numerical method or other previous algorithms.

Keywords: orthogonal functions, transforms, nonlinear distributed parameter system, two dimensional approximation and transforms

Introduction

In general, all physical systems have nonlinear characteristics of one form or another, control theory for nonlinear (distributed parameter) systems has been developed steadily. For instance, we may find that some control devices have appropriate nonlinear characteristics or nonlinear properties.[1][2] A number of techniques using orthogonal functions and its transforms have been proposed to solve the problems related to system analysis, system design and optimal control. The basic theory of orthogonal transform is to convert partial differential equation into an algebraic and operational matrices of integration are applied to simplify problems. Tzafestas and Stavroulakis introduced Walsh operational matrices and transform and Sibnha applied the double Walsh series for solving analysis of nonlinear distributed parameter system.[3][4] But application of this method is limited to a first order partial differential equation. In this paper, a method for optimal analysis and design of nonlinear distributed parameter system via orthogonal functions and transforms is suggested. For this, Walsh functions, block pulse functions and rationalized Haar functions are applied and two dimensional algorithm and approximating method by orthogonal transforms is introduced to solve partial differential equation.

Orthogonal Functions and Transform

1. Discrete Walsh Transforms

The Rademacher functions are a set of square waves for $t \in [0, 1)$, of unit height and repetition rate equal to 2^m, which can be generated by a BCD counter. The Walsh functions constitute a complete set of two values orthonormal functions $\Phi_k(t)$, $k=0, 1, 2, \ldots n-1$, $n=2^m$ in the interval (0, 1), and they can be defined in the several equivalent ways.

The set of Walsh functions is generally classified into three groups. The three type of Walsh orderings are a) Walsh ordering, b) Paley ordering and c) Hadamard ordering. The discrete Walsh transforms has found applications in many areas, including signal processing, pattern recognition and digital control systems.[5] Every function $f(t)$ which is integral is capable of being represented by Walsh series defined over the open interval (0, 1) as,

$$x(t)=a_0+a_1 Wal(1,t)+a_2 Wal(2,t)+\ldots.. \quad (1.1)$$

where coefficients are given by

$$a_k = \int_0^1 f(t)Wal(k,t)dt$$
(1.2)

From this we are able to define a transform pair as,

$$f(t) = \sum_{k=0}^{\infty} F(k)Wal(k,t) \quad (1.3)$$

$$F(k) = \int_0^1 f(t)Wal(k,t)dt \quad (1.4)$$

The integration shown in equation (1.4) may then be replaced by summation, using the trapezium rule on N sampling points, x_i, and we can write the finite discrete Walsh transform pair as,

$$X_n = \frac{1}{N}\sum_{i=0}^{N-1} x_i Wal(n,t),\ \ n = 0,1,2,\dots..N-1 \quad (1.5)$$

$$x_i = \sum_{n=0}^{N-1} x_n Wal(n,i),\ \ i = 0,1,2,\dots..N-1 \quad (1.6)$$

Let f_n^* denotes sampling of $f(t)$

$$f_n^* = \sum_{i=0}^{m-1} F_i Wal(n,i),\ \ n = 0,1,\dots..m-1 \quad (1.7)$$

And *i-th* discrete coefficients are given by

$$F_i = \frac{1}{m}\sum_{n=0}^{m-1} Wal(n,t) f_n^*,\ \ i = 0,1,2,\dots..m-1 \quad (1.8)$$

From equation (1.7) and (1.8), we can write f_n^* as follows:

$$f_n^* = m\int_{\frac{n}{m}}^{\frac{n+1}{m}} f(t)dt \cong \frac{1}{2}[f\left(\frac{n}{m}\right) + f\left(\frac{n+1}{m}\right)] \quad (1.9)$$

$$f^* = [f_0^* \quad f_1^* \quad f_2^* \dots..f_{m-1}^*]^T \quad (1.10)$$

$$F = [F_0 \quad F_1 \quad F_2 \dots..F_{m-1}]^T \quad (1.11)$$

2. Block Pulse Function Transforms

The block pulse functions are orthogonal with each other in the interval $t\in[0, T)$:

$$\int_0^{t_f} BF_i(t)\,dt = \begin{cases} h, & i=j \\ 0, & i\neq j \end{cases} \quad (1.12)$$

where $i, j = 1, 2, \dots.m$. The orthogonal property of block pulse functions is the basis of expanding functions into their block pulse series.[6] An arbitrary real bounded function $f(t)$, which is square integrable in the interval $t\in[0, T)$, can be expanded into a block pulse series in the sense of minimizing the mean square error between $f(t)$ and its approximation:

$$f(t) \cong \hat{f}(t) = \sum_{i=0}^{m-1} F_i\, BF_i(t) \quad (1.13)$$

Where $\hat{f}(t)$ is the block pulse series of the original $f(t)$, and F_i is the block pulse coefficient with respect to the i*th* block pulse function $BF_i(t)$. As an example, we can expand $f(t)=t^2$ into its block pulse series in the interval $t\in[0, 1)$ with $m=8$.

$$F_i = \frac{1}{h}\int_0^1 f(t)\,BF_i(t)dt = \frac{m}{t_f}\int_{i\frac{t_f}{m}}^{(i=1)\frac{t_f}{m}} f(t)dt$$

$$= \frac{1}{2}\left[f\left(i\frac{t_f}{m}\right) + f((i+1)\frac{t_f}{m})\right] \quad (1.14)$$

In order to expand the integral of a function into its block pulse series, we first consider the integral of each single block pulse function $BF_i(t)$.[4] For the case of $t\in[0, ih)$, we have:

$$\int_0^t BF_i(t)\,dt = 0 \quad (1.15)$$

For the case of $t\in[ih, (i+1)h)$, we have:

$$\int_0^t BF_i(t)\,dt = \int_0^{ih} BF_i(t)\,dt + \int_{ih}^t BF_i(t)\,dt$$

$$= t - ih \quad (1.16)$$

And in the case of $t\in[(i+1)h, t_f)$, *we have:*

$$\int_0^t BF_i(t)\,dt = \int_0^{ih} BF_i(t)\,dt + \int_{ih}^{(i+1)h} BF_i(t)\,dt + \int_{(i+1)h}^t BF_i(t)\,dt = h \quad (1.17)$$

From the above discussion, the block pulse series of the integral of all the m block pulse functions can be written in a compact form.[7]

$$\int_0^t BF_i(t)\,dt \cong PBF_i(t) \quad (1.18)$$

where the operational matrix P:

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2\cdots 2 \\ 0 & 1 & 2\cdots 2 \\ 0 & 0 & 1\cdots 2 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0\cdots 1 \end{bmatrix} \quad (1.19)$$

Figure 1 is the integral of block pulse functions with $m=4$.

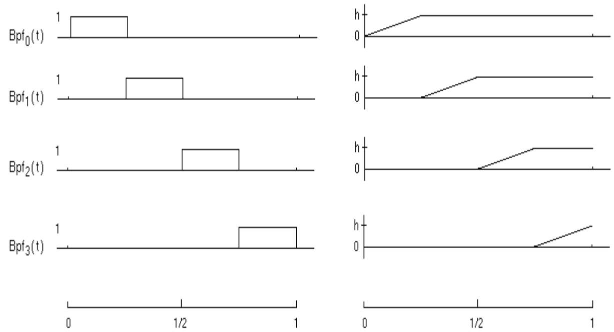

Figure 1. Integral of block pulse functions *(m=4)*

3. Rationalized Haar Transforms

The Haar function set forms a complete set of orthogonal rectangular functions such as Walsh and blocks pulse functions. But the Haar functions have some disadvantages of calculation because of including irrational numbers such as $\pm\sqrt{2^p}$, $(p = 1, 2, ..)$. The Rationalized Haar functions were introduced by M. Ohkita to overcome these disadvantages. The Rationalized Haar functions constitute of rational numbers only.[8]

$$RH(0,t) = 1 \quad for \;\; 0 \le t \le 1$$
(1.20)

$$RH(k,t) = \begin{cases} +1 \quad for \quad \frac{n}{2^p} \le t \le \frac{n+\frac{1}{2}}{2^p} \\ -1 \quad for \;\; (n+\frac{1}{2})/2^p \le t \le (n = 1)/2^p \\ 0 \quad for \quad elsewhere \end{cases}$$
(1.21)

where $k=2^p+n$, $p=0, 1, 2...\log_2 \frac{m}{2}$, $n=0, 1...2^p-1$

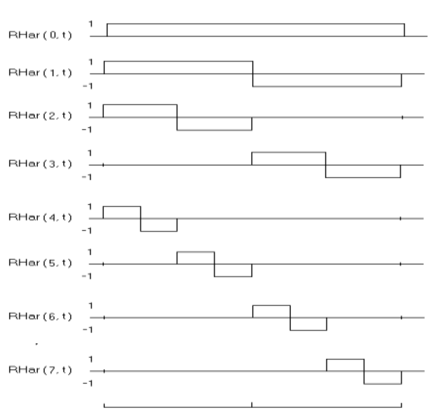

Figure 2. The first eight Rationalized Haar functions

The Rationalized Haar functions *RH(t)* is closed set. Thus, every signal *f(t)* which is absolutely integral in $t \in [0, 1)$ can be expanded in an infinite series of the Rationalized Haar functions and transform. The rationalized Haar functions are shown in figure 2.

$$f(t) = \sum_{k=0}^{\infty} f_k RH(k,t)$$
(1.22)

where f_k is the *k*th sequentially ordered coefficient of the Rationalized Haar functions expansion of function *f(t)* and *RH(k,t)* is the *k*th ordered Rationalized Haar functions. Now, we can get the approximation of *f(t)* as a finite series of the Rationalized Haar transform and matrix.

$$f(t) = \sum_{k=0}^{m-1} f_k RH(k,t) = F^T \emptyset_R(t)$$
(1.23)

where *F* is coefficient vector of *f(t)*, $\emptyset_R(t)$ is its Rationalized Haar functions vector and T denotes transposition.

$$F = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{m-1} \end{bmatrix} \qquad \emptyset_R(t) = \begin{bmatrix} RH(0,t) \\ RH(1,t) \\ \vdots \\ RH(m-1,t) \end{bmatrix} \tag{1.24}$$

If the function is given in the form of data, discrete transform method of the Rationalized Haar functions is applied as shown in equation (1.25).

$$f_i^* = \sum_{k=0}^{m-1} RH(k,i) F_i \tag{1.25}$$

where *i=0, 1, 2...m-1*

Equation (1.26) is matrix expressions of equation (1.25).
T denotes transpose of a matrix.

$$f_i^* = \emptyset_{Ri}^T F_i \tag{1.26}$$

$$f_i^* = \begin{bmatrix} f_0^* \\ f_1^* \\ \vdots \\ f_{m-1}^* \end{bmatrix}$$

$$\emptyset_{Ri} = \begin{bmatrix} RH(0,0) & RH(0,1) & & RH(0,m-1) \\ RH(1,0) & RH(1,1) & \cdots & RH(1,m-1) \\ \vdots & \vdots & & \vdots \\ RH(m-1,0) & RH(m-1,1) & & RH(m-1,m-1) \end{bmatrix}$$

$$F_i = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{m-1} \end{bmatrix} \tag{1.27}$$

Two Dimensional Approximations Algorithm

To solve partial differential equation of nonlinear distributed parameter system, two dimensional approximations algorithm that is based on the orthogonal transforms is applied. This method is useful to handle partial differential equation of two variables function. Consider a function *f(x,t)* of two variables on $x \in [0, 1]$ and $t \in [0, 1]$. Then, the orthogonal transforms can approximately represent. TDT denotes two dimensional transforms of the orthogonal functions.

$$f(x,t) = \sum_{i=0}^{\infty} f_i(x) TDT_i(t) \quad (2.1)$$

We can approximate equation (2.1) as equation (2.2). And using the orthogonal property, the coefficient functions $f_i(x)$ of equation (2.2) is obtained as equation (2.3).

$$f(x,t) = \sum_{i=0}^{n-1} f_i(x) TDT_i(t) \quad (2.2)$$

$$f_i(x) = \int_0^1 f(x,t) TDT_i(t) dt \quad (2.3)$$

The two dimensional approximation using orthogonal transforms of $f_i(t)$ gives,

$$f_i(t) = \sum_{j=0}^{m-1} TDT_j(x) f_{ji} \quad (2.4)$$

where $TDT_j(x)$ are the orthogonal functions with respect to x and f_{ji} is coefficients of two dimensional transforms. The coefficients f_{ji} is obtained by,

$$f_{ji} = \int_0^1 TDT_j(x) f_i(x) dx = 0101 fx,tTDTitTDTj(t)dxdt \quad (2.5)$$

Therefore two variables function *f(x,t)* cab be approximated as equation (2.6) using two dimensional orthogonal transforms.

$$f(x,t) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} f_{ji} TDT_i(t)\, TDT_j(x) = TDT_m^T(x) F_{mn} TDT_n(t) \quad (2.6)$$

Approximation of Nonlinear Distributed Parameter System

For a nonlinear system, consider the following partial differential equation model.[9][10]

$$c_5 \frac{\partial^2 y^{p5}(x,t)}{\partial t^2} + c_4 \frac{\partial^2 y^{p4}(x,t)}{\partial x \partial t} + c_3 \frac{\partial^2 y^{p3}(x,t)}{\partial x^2} + c_2 \frac{\partial^2 y^{p2}(x,t)}{\partial t} + c_1 \frac{\partial y^{p1}(x,t)}{\partial x} + c_0 y^{p0}(x,t) = u^{p6}(x,t) \quad (3.1)$$

To identify equation (3.1), the previous two dimensional algorithm can be applied. Now we can integrate twice the equation (3.1) with respect to *x* and *t*, and then we can get equation (3.2).

$$c_5 \int_0^x \int_0^x y^{p5}(x,t) dx dx + c_4 \int_0^x \int_0^t y^{p4}(x,t) dt dx + c_3 \int_0^t \int_0^t y^{p3}(x,t) dt dt + c_2 \int_0^x \int_0^x \int_0^t y^{p2}(x,t)\, dt dx dx$$

$$c_1 \int_0^x \int_0^x \int_0^t y^{p1}(x,t)\, dt dx dx$$

$$+ c_0 \int_0^x \int_0^x \int_0^t \int_0^t y^{p0}(x,t)\, dt dt dx dx + \int_0^x \int_0^x \int_0^t \alpha(x)\, dt dx dx$$

$$+ \int_0^x \int_0^x \int_0^t \beta(x)\, dt dx dx + \int_0^x \int_0^x \int_0^t \gamma(x)\, dt dx dx$$

$$- c_5 \int_0^x \int_0^x y^{p5}(x,0) dx dx - c_3 \int_0^t \int_0^t y^{p3}(0,t) dt dt$$

$$= \int_0^x \int_0^x \int_0^t \int_0^t u^{p6}(x,t)\, dt dt dx dx \quad (3.2)$$

where

$$y^{pz}(x,t) = TDT_m^T(x) Q_z TDT_n(t) \quad (3.3)$$

$$u^{pz}(x,t) = TDT_m^T(x) R_z TDT_n(t) \quad (3.4)$$

Q_z is zth column of

$$\mathrm{Q} = \sum_{i=1}^{m} \rho_{n-1}^{(m)} Y_{mn} \rho_{z-1}^{(n)} Y_i^T \quad (3.5)$$

where

$$\rho_i^{(m)} = \begin{bmatrix} \rho_i^{(\frac{m}{2})} & 0^{(\frac{m}{2})} \\ 0^{(\frac{m}{2})} & \rho_i^{(\frac{m}{2})} \end{bmatrix} \quad (3.6)$$

$$\alpha(x) = \sum_{i=0}^{k-1} \alpha_i TDT_m^T(x) F_{I+1,1} H_n(t),\ k \le m \quad (3.7)$$

$$\beta(x) = \sum_{i=0}^{l-1} \beta_i TDT_m^T(x) F_{I+1,1} H_n(t),\ l \le m \quad (3.8)$$

$\gamma(x) = \sum_{i=0}^{r-1} \gamma_i T D T_m^T(x) F_{I+1,1} H_n(t),\ r \le m$ (3.9)

Where F_{ij} is a matrix having the *(i,j)*th element unity and the remaining elements zero. The presented two dimensional approximations algorithm can be applied to the case of high order partial differential equation model and this approach is improved and advanced method than conventional methods.

Simulations

1. Example (a)

To illustrate the theoretical results consider a nonlinear system described by

$$\frac{dy(t)}{dt} = A_0 y(t) + A_1 y(t) u_1(t) + A_2 y(t) u_2(t) + Bu(t)$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad (4.1)$$

We assume that input $u_1(t)= e^{-0.5t}$, $u_2(t)=e^{-t}$ and initial condition $y(0)=0$. For simulation purpose forth order Runge-Kutta method with $\Delta=0.125$ is applied and its numerical simulation results for A_0, A_1, A_2 and B are as follows:

$$A_0 = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 0.6 & 0.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0.2 & 0.3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad (4.2)$$

Also, we can define indeterminate parameters using suggested optimal analysis method easily. The results are as follows:

$$A_0 = \begin{bmatrix} 0.0046 & 0.9837 \\ 0.4986 & -0.9956 \end{bmatrix}, A_1 = \begin{bmatrix} 0.0134 & 1.0246 \\ 0.4967 & 0.1935 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.9929 & -0.0160 \\ 0.1998 & 0.3014 \end{bmatrix}, B = \begin{bmatrix} 2.0230 & 0.9680 \\ 0.9925 & 1.0030 \end{bmatrix} \quad (4.3)$$

And the simulation results are shown in Table 1.

Table 1. Simulation data for the example

time	output		time	output	
	y_1	y_2		y_1	y_2
[0.000, 0.125]	0.41224	0.23479	[1.125 1.250]	5.49394	0.77072
[0.125, 0.250]	0.89079	0.43767	[1.250, 1.375]	5.98698	0.66186
[0.250, 0.375]	1.42305	0.60651	[1.375, 1.500]	6.43408	0.52824
[0.375, 0.500]	1.99437	0.73950	[1.500, 1.625]	6.83119	0.37374
[0.500, 0.625]	2.59060	0.83545	[1.625, 1.750]	7.17571	0.20241
[0.625, 0.750]	3.19778	0.89386	[1.750, 1.875]	7.46624	0.01830
[0.750, 0.875]	3.80283	0.91506	[1.875, 2.000]	7.70254	0.17460
[0.875, 1.000]	4.39388	0.90021	[2.000, 2.125]	7.88528	0.37249
[1.000, 1.125]	4.96052	0.85122	[2.125, 2.250]	8.01594	0.57179

2. Example (b)

Following nonlinear distributed parameter system is considered. Given a record of $y(x,t)$ and $u(x,t)$, the problem is to determine parameters of the system. For this purpose, $c_1=2$, $c_2=2$, $c_3=1$ and $m=n=4$ are taken.

$$c_1 \frac{\partial y(x,t)}{\partial x} + c_2 \frac{\partial y^2(x,t)}{\partial t} + c_3 y(x,t) = u(x,t) \quad (4.4)$$

$$y(x,0) = y(0,t) = 0, y(1,t) = 1 \quad (4.5)$$

And input/output functions are as follows:

$$u(x,t) = 4x^2 t + 2t + xt, y(x,t) = xt \quad (4.6)$$

To apply the proposed method, we integrate equation (4.4) with respect x and t and then expand the integrating results using two dimensional approximations transforms. Then we can convert the partial differential equation to the following algebraic equation (4.8) simply.

$$c_1 \int_0^x y(x,t)dx + c_2 \int_0^x y^2(x,t)dx$$

$$+ c_3 \int_0^t \int_0^x y(x,t)dxdt = \int_0^t \int_0^x u(x,t)dxdt \quad (4.7)$$

$$c_1 P_m^T Y_{mn} + c_2 Q_2 P_m + c_3 P_m^T Y_{mn} P_n = P_m^T U_{mn} P_n \quad (4.8)$$

Conclusions

This paper presents quite simple and accurate algorithm for optimal analysis and design of nonlinear distributed parameter system using orthogonal transform. For solving this problem, two dimensional approximations algorithm and orthogonal transforms is introduced. Applying the proposed method, a partial differential equation can be converted into an algebraic equation. Problems with unknown conditions pose no additional difficulties and can be handled in simple way. Since Walsh functions, block pulse functions or rationalized Haar functions are relatively easy to handle in computation, the proposed method for analysis and design of nonlinear control system is quite simple to implement and has, therefore, obvious advantages in practical situations. Example indicates that proposed method provides a more efficient and convenient procedures.

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